

Supersymmetric Deformations of Maximally Supersymmetric Gauge Theories. I

M. V. Movshev

Stony Brook University

Stony Brook, NY 11794-3651, USA

A. Schwarz*

Department of Mathematics

University of California

Davis, CA 95616, USA

October 4, 2009

Contents

1	Introduction	2
2	Preliminaries	9
2.1	Basic algebras	9
2.2	Calculation of Lie algebra cohomology and Hochschild cohomology.	13
2.3	The group $\text{Spin}(10, \mathbb{C})$ and the space of pure spinors	16
2.4	Euler characteristics	17
3	Infinitesimal SUSY Deformations	20

*The work of both authors was partially supported by NSF grant No. DMS 0505735 and by grants DE-FG02-90ER40542 and PHY99-0794

4	Homological Approach to Infinitesimal Deformations	29
5	Calculation of Cohomology	32
6	BV	42
7	Formal SUSY deformations	55
A	L_∞ and A_∞ algebras	59
B	Duality	73
C	On the relation of the Lie algebra and BV approaches to deformation problem	77
D	L_∞ action of the supersymmetry algebra in the BV formulation	81
E	Calculation of hypercohomology	84

Abstract

We study supersymmetric and super Poincaré invariant deformations of ten-dimensional super Yang-Mills theory and of its reduction to a point. We describe all infinitesimal super Poincaré invariant deformations of equations of motion and prove that all of them are Lagrangian deformations and all of them can be extended to formal deformations. Our methods are based on homological algebra, in particular, on the theory of L -infinity and A -infinity algebras. The exposition of this theory as well as of some basic facts about Lie algebra homology and Hochschild homology is given in appendices.

1 Introduction

The superspace technique is a very powerful tool of construction of supersymmetric theories. However this technique does not work for theories with large number of supersymmetries. It is possible to apply methods of homological

algebra and formal non-commutative geometry to prove existence of supersymmetric deformations of gauge theories and give explicit construction of them. In this paper we discuss results obtained by such methods in the analysis of SUSY deformations of 10-dimensional SUSY YM-theory (SYM theory) and its dimensional reductions.

These deformations are quite important from the viewpoint of string theory. It is well known that D-brane action in the first approximation is given by dimensional reduction of ten-dimensional SYM theory; taking into account the α' corrections we obtain SUSY deformation of this theory. (More precisely, we obtain a power series with respect to α' specifying a formal deformation of the theory at hand.)

Our approach is closely related to pure spinors techniques; it seems that it could be quite useful to understand better the pure spinor formalism in string theory constructed by Berkovits [3].

Recall that in component form the action functional of 10-dimensional SUSY YM-theory looks as follows:

$$S_{SYM}(A, \chi) = \int \mathcal{L}_{SYM} d^{10}x = \int \text{tr} \left(\frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} \Gamma_{\alpha\beta}^i \chi^\alpha \nabla_i \chi^\beta \right) d^{10}x \quad (1)$$

where $A_i(x)$ are gauge fields with values in the Lie algebra of the unitary group $U(N)$, $\nabla_i = \frac{\partial}{\partial x_i} + A_i(x)$ are covariant derivatives, χ^α are chiral spinors with values in the adjoint representation, $F_{ij} = [\nabla_i, \nabla_j]$ is the curvature.¹

We consider deformations that can be described by action functionals of the form

$$\int \text{tr}(Y) d^{10}x \quad (2)$$

where $\text{tr}(Y)$ is an arbitrary gauge invariant local expression in terms of gauge fields A_i and spinor fields χ^α . Here Y involves arbitrary product of covariant derivatives of the curvature F_{ij} and spinor fields χ^α . One can say that Y is gauge covariant local expression. The integrals in formulas (1) and (2) are understood

¹In this text by default small Roman indices i, j run over $1, \dots, 10$, Greek indices α, β, γ run over $1, \dots, 16$. We do not distinguish lower and upper Roman indices because we assume that the ten-dimensional space is equipped with the Riemann metric $(dx^i)^2$.

as formal expressions. We completely ignore the issues of convergence. In this formal approach the integrals are invariant with respect to some field transformation iff the variation of the integrand is a total derivative. We consider only deformations that can be applied simultaneously to gauge theories with all gauge groups $U(N)$ where N is an arbitrary positive integer. This remark is important because it is very likely that we miss some important deformations that are defined for a finite range of N .

It is also interesting to consider the dimensional reductions of 10-D SUSY YM theory; after reducing to dimension 4 we obtain N=4 SUSY YM theory; reducing to dimension one leads to BFFS matrix model, reducing to dimension 0 leads to IKKT matrix model.

Of course, reducing a deformation of 10-D SUSY YM-theory we obtain a deformation of the corresponding reduced theory. However the reduced theory can have more deformations. We will give a complete description of SUSY-deformations of 10-D SUSY YM theory and its reduction to D=0 (of IKKT model).

In the components the supersymmetry operators θ_α are equal to

$$\begin{aligned}\theta_\alpha \nabla_i &= \Gamma_{\alpha\beta i} \chi^\beta \\ \theta_\alpha \chi^\beta &= \Gamma_{\alpha}^{\beta ij} F_{ij}\end{aligned}\tag{3}$$

Denote by D_i the lift of the space-time translation $\partial/\partial x^i$ to the space of the gauge fields and spinor fields. The lift is defined only up to gauge transformation. We fix the gauge freedom in a choice of D_i requiring that

$$\begin{aligned}D_i \nabla_j &= F_{ij} \\ D_i \chi^\alpha &= \nabla_i \chi^\alpha\end{aligned}\tag{4}$$

For fields obeying the equations of motion of S_{SYM} infinitesimal symmetries θ_α

satisfy

$$\begin{aligned}
[\theta_\alpha, \theta_\beta] &= \Gamma_{\alpha\beta}^i D_i \\
[\theta_\alpha, D_i] A_k &= -\Gamma_{\alpha\beta i} \nabla_k \chi^\beta \\
[\theta_\alpha, D_i] \chi^\gamma &= \Gamma_{\alpha\beta i} [\chi^\beta, \chi^\gamma]
\end{aligned} \tag{5}$$

We see that on shell (on the space of solutions of the equations of motion where gauge equivalent solutions are identified) supersymmetry transformations commute with space-time translations:

$$[\theta_\alpha, D_i] = 0 \text{ on shell.} \tag{6}$$

Talking about SUSY-deformations we have in mind deformations of action functional and simultaneous deformation of these 16 supersymmetries.

Notice that 10-D SUSY YM-theory has also 16 trivial supersymmetries, corresponding to constant shifts of fermion fields. The analysis of deformations preserving these symmetries was left out of scope of the present paper.

We will work with Lagrangian densities \mathcal{L} instead of action functionals $S = \int \mathcal{L} d^{10}x$.

As a first approximation to the problem we would like to solve we will study infinitesimal supersymmetric (SUSY) deformations of equations of motion of ten-dimensional SUSY Yang-Mills theory. We reduce this problem to a question in homological algebra. The homological reformulation leads to highly nontrivial, but solvable problem. We will analyze also super Poincaré invariant (= supersymmetric + Lorentz invariant) infinitesimal deformations. We will prove that all of them are Lagrangian deformations of equations of motion (i.e. the deformed equations come from deformed Lagrangian).

One of the tools that we are using is the theory of A_∞ and L_∞ algebras. The theory of L_∞ algebras is closely related to BV formalism. One can say that the theory of L_∞ algebras with invariant odd inner product is equivalent to classical BV-formalism if we are working at formal level. (This means that we are considering all functions at hand as formal power series). The theory of

A_∞ algebras arises if we would like to consider Yang-Mills theory for all gauge groups $U(N)$ at the same time.

Recall that in classical BV-formalism the space of solutions to the equations of motion (EM) can be characterized as zero locus Sol of odd vector field Q obeying $[Q, Q] = 0$.² It is convenient to work with the space Sol/\sim obtained from zero locus Sol after identification of physically equivalent solutions.

One can consider Q as a derivation of the algebra of functionals on the space of fields M . The space M is equipped with an odd symplectic structure; Q preserves this structure and therefore the corresponding derivation can be written in the form $Qf = \{S, f\}$ where $\{\cdot, \cdot\}$ stands for the odd Poisson bracket and S plays the role of the action functional in the BV formalism.

A vector field q_0 on M is an infinitesimal symmetry of EM if $[Q, q_0] = 0$. However, studying the symmetry Lie algebra we should disregard trivial symmetries (symmetries of the form $q_0 = [Q, \rho_0]$). Hence, in BV formalism talking about symmetry Lie algebra \mathfrak{g} with structure constants $f_{\tau_1 \tau_2}^{\tau_3}$ we should impose the condition

$$[q_{\tau_1}, q_{\tau_2}] = f_{\tau_1 \tau_2}^{\tau_3} q_{\tau_3} + [Q, q_{\tau_1 \tau_2}] \quad (7)$$

on the infinitesimal symmetries q_τ . In this case index τ labels a basis in the space of symmetries. We say in this case that \mathfrak{g} acts weakly on the space of fields. However, it is more convenient to work with notion of L_∞ action of \mathfrak{g} . To define L_∞ action we should consider in addition to $q_\tau, q_{\tau_1 \tau_2}$ also their higher analogs $q_{\tau_1, \dots, \tau_k}$ and impose some conditions generalizing (7). Introducing the generating function q we can represent these conditions in compact form:

$$d_{\mathfrak{g}} q + [Q, q] + \frac{1}{2}[q, q] = 0.$$

Here

$$d_{\mathfrak{g}} = \frac{1}{2} f_{\tau_1 \tau_2}^{\tau_3} c^{\tau_1} c^{\tau_2} \frac{\partial}{\partial c^{\tau_3}} \quad (8)$$

stands for the differential calculating the Lie algebra cohomology of \mathfrak{g} , c^τ are ghosts corresponding to the Lie algebra. This equation can be formulated also

²We use a unified notation $[\cdot, \cdot]$ for the commutators and super-commutators.

in Lagrangian BV formalism; then we should replace the supercommutators of vector fields by odd Poisson bracket of functionals depending of fields, antifields, ghosts and antifields for ghosts.

Using the equation (7) we can study the problem of classification of deformations preserving the given Lie algebra of symmetries. It is important to emphasize that we can start with an arbitrary BV formulation of the given theory and the answer does not depend on our choices. In the case of infinitesimal deformations the classification can be reduced to a homological problem (to the calculation of cohomology of the differential $d_{\mathfrak{g}} + [q, \cdot]$ acting on the space of vector fields depending on ghosts).

The present paper consists of two parts. In the first part we apply the above ideas to the ten-dimensional SYM theory and to its reduction to a point. We describe in this language all infinitesimal super Poincaré invariant deformations. We show that almost all of them are given by a simple general formula. We sketch the proof of the fact that SUSY infinitesimal deformations can be extended to formal SUSY deformations (by definition a formal deformation is a deformation that can be written as a formal power series with respect to some parameter; in string theory the role of this parameter is played by α').

The paper will be organized in the following way: Preliminaries (Section 2) contains some mathematical information needed in our constructions and proofs. It is reasonable to skip this section and start reading with Section 3 returning to Section 2 as necessary. In Section 3 we give a complete description of infinitesimal SUSY deformations. We give a very explicit formula that works for almost all deformations.³ In Section 7 we prove that all of infinitesimal SUSY deformations can be extended to formal deformations. In Section 4 we reduce the computation of the infinitesimal SUSY deformations to a homological problem. In Section 5 and in Appendix E we sketch the solution of this problem. In Section 6 we approach to the problem of infinitesimal deformations from the point of view of BV formalism. This approach leads to another homological

³In the Part II we show that exceptional deformations are related to the homology of SUSY Lie algebra.

formulation of our problem.

In Appendix C we relate this formulation to formulation of Section 4. The approach based on BV formalism works in more general situation.

The reader who is more interested in methods rather in concrete results can start reading beginning with Appendices A,B containing brief exposition of the theory of L_∞ and A_∞ algebras and of duality of differential associative algebras, that play an important role in our calculations.

In the second part of the paper we describe results about deformations of d -dimensional reduction of ten-dimensional SYM theory for the case when d is an arbitrary integer between 0 and 10 generalizing the results obtained in the first part for $d = 0$ and $d = 10$. In this part we give a complete calculation of Euler characteristics of all relevant cohomology groups and use this calculation to make a conjecture about the structure of these cohomology groups. For the cases $d = 0$ and $d = 10$ one can prove this conjecture. We show that the homology of the supersymmetry Lie algebras are related to supersymmetric deformations and analyze these homologies.

The present paper concludes the series of papers devoted to the analysis of deformations of SYM theories [18], [19], [16], [17], [20]. It contains a review of most important results of these papers as well as some new constructions. The paper can be read independently of other papers of the series, but we refer to these papers for some complicated proofs (and in the opposite case when the proofs are simple and we feel that there is no necessity to repeat them).

Acknowledgment

The first author wishes to thank Max-Planck-Institut für Mathematik in den Naturwissenschaften in Leipzig, where part of this work has been done for hospitality and excellent working conditions.

The second author is thankful to Max-Planck-Institut für Mathematik in Bonn and Institut des Hautes études Scientifiques for stimulating mathematical atmosphere in which part of this work has been done.

Both authors benefited from conversations with N.Berkovits, M.Kontsevich, N.Nekrasov and D.Sullivan.

2 Preliminaries

2.1 Basic algebras

In this section we describe following [18] some algebras related to SYM theory and duality relations between these algebras.

Let us define the algebra $\mathcal{S} = \bigoplus_{k \geq 0} \mathcal{S}_k$ as an algebra with generators $\lambda^1, \dots, \lambda^{16}$ and relations

$$\Gamma_{\alpha\beta}^i \lambda^\alpha \lambda^\beta = 0, \quad (9)$$

where $\Gamma_{\alpha\beta}^i$ are ten-dimensional Γ -matrices. We can interpret \mathcal{S} as an algebra of polynomial functions on the space $C\mathcal{Q}$ of pure spinors (spinors obeying $\Gamma_{\alpha\beta}^i \lambda^\alpha \lambda^\beta = 0$); then \mathcal{S}_k a space of polynomial functions of degree k . We can define \mathcal{S} in terms of space \mathcal{Q} obtained from $C\mathcal{Q} \setminus 0$ by means of identification of proportional spinors. Then \mathcal{S}_k can be identified with the space of holomorphic sections of line bundle $\mathcal{O}(k)$ over \mathcal{Q} .

The reduced Berkovits algebra B_0 is a differential graded commutative algebra. It is generated by even λ^α obeying pure spinor relations (9) and odd ψ^α . The differential d satisfies $d(\psi^\alpha) = \lambda^\alpha, d(\lambda^\alpha) = 0$.

One can also give a description of B_0 in terms of functions on $C\mathcal{Q}$. Its elements are polynomial functions depending on pure spinor λ and odd spinor ψ . We can interpret ψ^α as coordinates on odd spinor space ΠS . The differential is represented by the odd vector field $\lambda^\alpha \frac{\partial}{\partial \psi^\alpha}$.

The (unreduced) Berkovits algebra B can be defined as the algebra of complex-valued functions of pure spinor λ , odd spinor ψ and $x \in \mathbb{R}^{10}$ that depend polynomially of $\lambda \in C\mathcal{Q}$. The differential is defined as the derivation

$$\lambda^\alpha \left(\frac{\partial}{\partial \psi^\alpha} + \Gamma_{\alpha\beta}^i \psi^\beta \frac{\partial}{\partial x^i} \right). \quad (10)$$

The algebras \mathcal{S}, B_0, B are quadratic algebras, i.e. they are described by generators obeying quadratic relations.

Let us consider an arbitrary unital quadratic algebra A , with generators w_1, \dots, w_n , obeying quadratic relations $r_k = r_k^{ij} w_i w_j = 0$. In more invariant

terms we say that A is generated by linear space $W = \langle w_1, \dots, w_n \rangle$. Relations generate a linear subspace $R \subset W \otimes W$. By construction the algebra A is graded $A = \bigoplus_{i \geq 0} A_i$ where $A_0 = \mathbb{C}, A_1 = W$. We shall use the following convention for grading indices

$$N_i = N^{-i}.$$

Let us define the dual quadratic algebra $A^! = \bigoplus_{i \geq 0} A_i^!$. As an algebra it is generated by dual linear space $W^* = \langle w^{*1}, \dots, w^{*n} \rangle$, where $\langle w^{*i}, w_j \rangle = \delta_j^i$. It has relations corresponding to the subspace $R^\perp \subset W^* \otimes W^*$. In other words R^\perp has a basis s^m which corresponds to a basis in the space of solutions of linear system $\sum_{ij} r_k^{ij} s_{ij} = 0$.

The duality of quadratic algebras has some good properties in case of Koszul algebras that will be defined below. We consider a subspace $A_1 \otimes A_1^! = W \otimes W^*$ of the tensor product $A \otimes A^!$. Let

$$e = \sum_i w_i \otimes w^{*i} \in W \otimes W^* \quad (11)$$

denote the canonical tensor corresponding to the matrix of identity transformation. It is easy to see that $e^2 = 0$. One can use e to define a differential on any $A \otimes A^!$ -module K . Let us consider the module $K = A \otimes A^{!*}$. It contains a subspace $\mathbb{C} = A_0 \otimes A_0^{!*}$ which generates nontrivial subspace in cohomology $H(A \otimes A^{!*})$. We say that A is Koszul if this subspace exhausts the cohomology.

One of the properties of Koszul algebras is that A is Koszul iff $A^!$ is.

One can calculate dual algebras to \mathcal{S}, B_0, B .

The dual algebra to \mathcal{S} is the graded algebra $U(L)$ on generators $\theta_1, \dots, \theta_{16}$ of degree one, which satisfy relations

$$\Gamma_{i_1, \dots, i_5}^{\alpha\beta} [\theta_\alpha, \theta_\beta] = 0. \quad (12)$$

The algebra $U(L)$ is the universal enveloping of the graded Lie algebra $L = \sum L^n$ that is defined by the same relations. (Notice that the grading on L we are using and the grading of general theory in Appendix B differ. To compare these two gradings we note that in Appendix B the generators of L have degree minus one

and generators of \mathcal{S} have degree two. To switch between positive and negative gradings we use the convention $L_n = L^{-n}$.) It is easy to see that one can find a basis $\mathbf{D}_1, \dots, \mathbf{D}_{10}$ of L^2 obeying

$$[\boldsymbol{\theta}_\alpha, \boldsymbol{\theta}_\beta] = \Gamma_{\alpha\beta}^i \mathbf{D}_i. \quad (13)$$

The algebra $B_0^!$ dual to B_0 is the tensor product $U(L) \otimes \mathbb{C}[s_1, \dots, s_{16}]$. The differential acts by the formula $d(\theta_\alpha) = s_\alpha$. In other words the algebra $B_0^!$ is a universal enveloping of a direct sum $H = L + S$, where $S = H_0$ is an abelian Lie algebra in degree zero. We consider H as a differential Lie algebra with the differential d .

Let us introduce a Lie algebra YM with even generators $\mathbf{D}_1, \dots, \mathbf{D}_{10}$ and odd $\boldsymbol{\chi}^1, \dots, \boldsymbol{\chi}^{16}$ obeying relations

$$\begin{aligned} & \sum_{i=1}^{10} [\mathbf{D}_i, [\mathbf{D}_i, \mathbf{D}_m]] - \\ & - \frac{1}{2} \sum_{\alpha\beta=1}^{16} \Gamma_{\alpha\beta}^m [\boldsymbol{\chi}^\alpha, \boldsymbol{\chi}^\beta] = 0 \quad m = 1, \dots, 10 \end{aligned} \quad (14)$$

$$\sum_{\beta=1}^{16} \sum_{i=1}^{10} \Gamma_{\alpha\beta}^i [\mathbf{D}_i, \boldsymbol{\chi}^\beta] = 0 \quad \alpha = 1 \dots 16 \quad (15)$$

The relations coincide with equations of motion of D=10 SYM theory reduced to a point. One can say that N - dimensional representations of the algebra YM gives a classical solution of the reduced SYM theory (of IKKT model).

It is easy to construct a homomorphism of the Lie algebra YM into L (or, more precisely, into $\bigoplus_{k \geq 2} L^k$). Namely, we should send its generators into \mathbf{D}_i , defined by (13) and into $\boldsymbol{\chi}^\beta$ defined by the formula

$$\Gamma_{\alpha\beta i} \boldsymbol{\chi}^\beta = [\boldsymbol{\theta}_\alpha, \mathbf{D}_i] \quad (16)$$

Proposition 1 *The algebra YM is isomorphic to $\bigoplus_{k \geq 2} L^k$. The obvious map $\bigoplus_{k \geq 2} L^k \rightarrow (L + S, d)$ is a quasi-isomorphism. Similarly $U(YM)$ is quasi-isomorphic to $B_0^!$.*

Recall that a *homomorphism of differential algebras (modules)* is called a *quasi-isomorphism* if it induces isomorphism on homology.

The dual algebra to B is the universal enveloping of differential graded Lie algebra \tilde{L} , defined as a semi- direct product $L \ltimes \Lambda$, where Λ is an abelian Lie algebra with the generators $s_1, \dots, s_{16}, \varsigma_1, \dots, \varsigma_{10}$ of degree zero and one respectively. The nontrivial commutation relations between L and Λ are $[\theta_\alpha, s_\beta] = \Gamma_{\alpha\beta}^i \varsigma_i$. The differential acts by the formulas $d(\theta_\alpha) = s_\alpha, d(D_i) = \varsigma_i$.

We define Lie algebra $TYM \subset L$ as

$$TYM = \bigoplus_{i \geq 3} L^i$$

It is clear that $F_{ij} = [D_i, D_j]$ and χ^α belong to TYM . Moreover, they generate TYM as an ideal of YM . More precisely, as an algebra TYM is generated by expressions $\nabla_{i_1} \dots \nabla_{i_n} \Phi$ where Φ is either F_{kl} or χ^α and $\nabla_i(x) = [D_i, x]$. In the framework of ten dimensional Yang-Mills theory we can interpret these expressions as covariant derivatives of field strength and spinor field. We can say that the elements of $U(TYM)$ are gauge covariant local expressions.

Proposition 2 *The obvious map $TYM \rightarrow (\tilde{L}, d)$ is a quasi-isomorphism. Similarly $U(TYM)$ is quasi-isomorphic to $B^!$.*

One can prove that all quadratic algebras we use are Koszul algebras.⁴

We have noticed that the $\mathcal{S}^!, B_0^!, B^!$ are universal envelopes of graded Lie algebras. This is a particular case of more general statement that $\mathcal{C}^!$ for quadratic graded commutative algebra \mathcal{C} is a universal envelope of a graded Lie algebra. This follows from the remark that relations in the dual algebra can be written

⁴The algebra L has the following geometric interpretation. It is a Lie subalgebra of the algebra of vector fields $Vect$ on the space $Sol = Sol_N$ of solutions of Yang-Mills equations with the gauge group $U(N)$. (Notice that we do not identify gauge equivalent solutions.) More precisely, this is a Lie subalgebra generated by supersymmetries. We have this inclusion because of the formulas (3) and (5).

The universal enveloping algebras $U(TYM)$, $U(YM)$ and $U(L)$ become associative subalgebras in algebras of differential operators $Diff$ on the space of solutions.

as relations between graded commutators. These relations can be considered as defining relations of a Lie algebra and also of its universal enveloping algebra.

2.2 Calculation of Lie algebra cohomology and Hochschild cohomology.

We will formulate some results that can be applied to calculate Lie algebra cohomology and Hochschild cohomology. (See Appendix A for the definition of Lie algebra cohomology and Hochschild cohomology and Appendix B for sketch of the proof.)

Let us consider a graded commutative Koszul algebra \mathcal{C} and its dual algebra $\mathcal{C}^! = U(\mathfrak{g})$ where \mathfrak{g} is a graded Lie algebra. Let N be a graded \mathfrak{g} -module (representation of \mathfrak{g}). The following statement can be used to simplify the calculation of Lie algebra cohomology.

Proposition 3 *The cohomology $H^\bullet(\mathfrak{g}, N)$ is equal to the cohomology of the complex $N_c \stackrel{\text{def}}{=} N \otimes \mathcal{C}$ (the \mathcal{C} -grading defines the cohomological grading in the tensor product). The differential d is defined by the element*

$$e = w^{*i} \otimes w_i. \quad (17)$$

where the elements of the basis $w_i \in \mathcal{C}^1$ act on \mathcal{C} by means of multiplication from the left and the action of the elements of the dual basis $w^{*i} \in (\mathcal{C}^1)^* \subset \mathcal{C}^!$ is defined by means of representation of \mathfrak{g} on N .

The subspaces $N_c^\bullet \mathbf{m} = \bigoplus_{i+j=\mathbf{m}} \mathcal{C}_i \otimes N_j$ are d -invariant.⁵ The complex N_c^\bullet coincides with the direct sum $\bigoplus_{\mathbf{m}} N_c^\bullet \mathbf{m}$.

The component $H^{k, \mathbf{m}}(\mathfrak{g}, N)$ of k -th cohomology group of homogeneity \mathbf{m} coincides with $H^k(N_c^\bullet \mathbf{m})$.

There exists a similar statement for Lie algebra homology. The complex $N_h^\bullet = N \otimes \mathcal{C}^*$ is the direct sum of subcomplexes $N_h^\bullet = \bigoplus_{\mathbf{m}} N_h^\bullet \mathbf{m}$. The homological

⁵To avoid a possible confusion of cohomological and internal homogeneous grading we reserve the bold index for the latter.

grading on $N_h^\bullet \mathbf{m}$ is defined as follows:

$$N_h^\bullet \mathbf{m} = (N_{m_0} \otimes \mathcal{C}_{m-m_0}^* \xrightarrow{d} \dots N_0 \otimes \mathcal{C}_m^* \xrightarrow{d} \dots N_{m-1} \otimes \mathcal{C}_1^* \xrightarrow{d} N_m \otimes \mathcal{C}_0^*) \quad (18)$$

Proposition 4 *The homological version of the last proposition is the isomorphism $H_\bullet(\mathfrak{g}, N) \cong H_\bullet(N \otimes \mathcal{C}^*)$ and its refinement*

$$H_{k,\mathbf{m}}(\mathfrak{g}, N) \cong H^{m-k}(N_h^\bullet \mathbf{m}).$$

Propositions 3,4 enable us to give a different description of $H^\bullet(L, N)$, $H_\bullet(L, N)$ for a graded L -module N .

Corollary 5 *The cohomology $H^\bullet(L, N)$ is equal to the cohomology of the complexes $N_c \stackrel{\text{def}}{=} N \otimes \mathcal{S}$. The differential is a multiplication by*

$$e = \lambda^\alpha \theta_\alpha. \quad (19)$$

The cohomological grading coincides with the grading of \mathcal{S} -factor. The total degree is preserved by d . The complex N_c splits according to degree:

$$N_c^\bullet \mathbf{m} = (N_m \otimes \mathcal{S}_0 \rightarrow N_{m+1} \otimes \mathcal{S}_1 \rightarrow \dots) \quad (20)$$

The complex $N_c^\bullet \mathbf{m}$ is defined for positive and negative \mathbf{m} , we assume that $N_m = 0$ if $m < m_0$. Then $H^{k,\mathbf{m}}(L, N) = H^k(N_c^\bullet \mathbf{m})$.

There is also a degree decomposition in homology $H_k(L, N) = \bigoplus_{\mathbf{m}} H_{k,\mathbf{m}}(L, N)$.

Corollary 6 *The homology $H_\bullet(L, N)$ is equal to the cohomology of the complex $N_h \stackrel{\text{def}}{=} N \otimes \mathcal{S}^*$. The space $\mathcal{S}^* = \bigoplus_{n \geq 0} \mathcal{S}_n^*$ is an \mathcal{S} -bimodule dual to \mathcal{S} . The differential is a multiplication by e (19). The homological degree coincides with the grading of \mathcal{S}^* -factor. The complex N_h splits :*

$$N_h^\bullet \mathbf{m} = N_{m_0} \otimes \mathcal{S}_{m-m_0}^* \xrightarrow{d} \dots N_0 \otimes \mathcal{S}_m^* \xrightarrow{d} \dots N_{m-1} \otimes \mathcal{S}_1^* \xrightarrow{d} N_m \otimes \mathcal{S}_0^* \quad (21)$$

$$\text{and } H_{k,\mathbf{m}}(L, N) = H^{m-k}(N_h^\bullet \mathbf{m}).$$

The Propositions 2,3 are particular cases of more general statements formulated in terms of Hochschild cohomology and homology (see Appendix A for definition.)

Proposition 7 *We assume that A is Koszul. The Hochschild cohomology $HH^\bullet(A, N)$ is equal to the cohomology of the complex $N_c \stackrel{\text{def}}{=} N \otimes A^!$. The differential is the graded commutator with e . The complex N_c splits according to degree:*

$$N_c^\bullet \mathbf{m} = N_m \otimes A_0^! \rightarrow N_{m+1} \otimes A_1^! \rightarrow \dots \quad (22)$$

The complex $N_c^\bullet \mathbf{m}$ is defined for positive and negative \mathbf{m} , we assume that $N_m = 0$ if $m < m_0$. Then $HH^{k,\mathbf{m}}(A, N) = H^k(N_c^\bullet \mathbf{m})$.

We sketch the proof of this proposition in Appendix B.

There is also a similar statement for Hochschild homology. We are using in this statement the degree decomposition $HH_k(A, N) = \bigoplus_{\mathbf{m}} HH_{k,\mathbf{m}}(A, N)$.

Proposition 8 *We assume that A is Koszul. Homology $HH_\bullet(A, N)$ are equal to the cohomology of the complex $N_h \stackrel{\text{def}}{=} N \otimes A^{!*}$. The space $A^{!*} = \bigoplus_{n \geq 0} A_n^{!*}$ is an $A^!$ -bimodule dual to $A^!$. The differential is a commutator with e given by the formula (11). The complex N_h splits :*

$$N_h^\bullet \mathbf{m} = N_{m_0} \otimes A_{m-m_0}^{!*} \xrightarrow{d} \dots N_0 \otimes A_m^{!*} \xrightarrow{d} \dots N_{m-1} \otimes A_1^{!*} \xrightarrow{d} N_m \otimes A_0^{!*} \quad (23)$$

$$\text{Then } HH_{k,\mathbf{m}}(A, N) = H^{m-k}(N_h^\bullet \mathbf{m}).$$

Propositions 3,4 follow from Propositions 7,8 if we set $A = U(\mathfrak{g})$, $A^! = \mathcal{C}$ and use the fact that Lie algebra cohomology of \mathfrak{g} with coefficients in a \mathfrak{g} -module coincide with Hochschild cohomology of $U(\mathfrak{g})$ with coefficients in $U(\mathfrak{g})$ -bimodule. ⁶

There is a similar isomorphism for homology.

⁶Let N be a $U(\mathfrak{g})$ -bimodule. Define a new structure of \mathfrak{g} -module on N by the formula $l \otimes n \rightarrow ln - nl, l \in \mathfrak{g}, n \in N$ There is an isomorphism

$$HH^i(U(\mathfrak{g}), N) \rightarrow H^i(\mathfrak{g}, N), \quad (24)$$

defined by the formula:

$$\gamma(l_1, \dots, l_n) \rightarrow \tilde{\gamma} = \frac{1}{n!} \sum_{\sigma \in S_n} \pm \gamma(l_{\sigma(1)}, \dots, l_{\sigma(n)}), l_i \in \mathfrak{g} \quad (25)$$

2.3 The group $\text{Spin}(10, \mathbb{C})$ and the space of pure spinors

The complex group $\text{Spin}(10, \mathbb{C})$ acts transitively on \mathcal{Q} ; the stable subgroup of a point is a parabolic subgroup P . To describe the Lie algebra \mathfrak{p} of P we notice that the Lie algebra $\mathfrak{so}(10, \mathbb{C})$ of $\text{SO}(10, \mathbb{C})$ can be identified with $\Lambda^2(V)$ (with the space of antisymmetric tensors ρ_{ab} where $a, b = 1, \dots, 10$). The vector representation V of $\text{SO}(10, \mathbb{C})$ restricted to the group $\text{GL}(5, \mathbb{C}) \subset \text{SO}(10, \mathbb{C})$ is equivalent to the direct sum $W \oplus W^*$ of vector and covector representations of $\text{GL}(5, \mathbb{C})$. The Lie algebra of $\text{SO}(10, \mathbb{C})$ as vector space can be decomposed as $\Lambda^2(W) + \mathfrak{p}$ where $\mathfrak{p} = (W \otimes W^*) + \Lambda^2(W^*)$ is the Lie subalgebra of \mathfrak{p} . Using the language of generators we can say that the Lie algebra $\mathfrak{so}(10, \mathbb{C})$ is generated by skew-symmetric tensors m_{ab}, n^{ab} and by k_a^b where $a, b = 1, \dots, 5$. The subalgebra \mathfrak{p} is generated by k_a^b and n^{ab} . Corresponding commutation relations are

$$\begin{aligned} [m, m'] &= [n, n'] = 0 \\ [m, n]_a^b &= m_{ac} n^{cb} \\ [m, k]_{ab} &= m_{ac} k_b^c + m_{cb} k_a^c \\ [n, k]_{ab} &= n^{ac} k_c^b + n^{cb} k_c^a \end{aligned} \tag{26}$$

There exists one-to-one correspondence between $\text{Spin}(10, \mathbb{C})$ -invariant holomorphic vector bundles over \mathcal{Q} and complex representations of P (lifting the action of the group on the base to the total space of vector bundle we obtain an action of stabilizer on the fiber). One-dimensional representation of P corresponding to the line bundle $\mathcal{O}(k)$ over \mathcal{Q} will be denoted μ_k ; it is easy to check that μ_k is a tensor product of k copies of μ_1 . The space of spinors can be embedded into Fock space \mathcal{F} (Fock representation of canonical anti-commutation relations $[a_i, a_j]_+ = 0, [a_i^*, a_j^*]_+ = 0, [a_i, a_j^*]_+ = \delta_{ij}$). The manifold \mathcal{Q} can be realized as the orbit of Fock vacuum with respect to the action of the group of linear canonical transformations (transformations preserving anti-commutation relations). For every vector $x \in \mathcal{F}$ we consider the subspace $W^*(x)$ of the space V of linear combinations $A = \sum \rho^i a_i + \sum \tau^j a_j^*$ obeying $Ax = 0$. For $x \in \mathcal{Q}$ the subspace $W^*(x)$ is five-dimensional. The subspaces $W^*(x)$ specify

a $\text{Spin}(10)$ -invariant vector bundle over \mathcal{Q} that will be denoted by \mathcal{W}^* ; corresponding representation of P will be denoted by W^* . The bundle over \mathcal{Q} having fibers $V/W^*(x)$ will be denoted by \mathcal{W} ; corresponding representation of P will be denoted by W . The group P contains a two-sheet cover $\widetilde{\text{GL}}(5)$ of $\text{GL}(5)$. The notations W and W^* for representations of P agree with notations for vector and covector representations of $\text{GL}(5)$.

Notice that $\text{Spin}(10)$ -representation contents of first two components of L is

$$L^1 = [0, 0, 0, 1, 0] \quad (27)$$

$$L^2 = [1, 0, 0, 0, 0] \quad (28)$$

$$L^3 = [0, 0, 0, 0, 1] \quad (29)$$

$$L^4 = [0, 1, 0, 0, 0] \quad (30)$$

$$\dots \quad (31)$$

There are the following identifications of $\widetilde{\text{GL}}(5)$ -representations :

$$L^1 = \mu_{-1} + \Lambda^2(W) \otimes \mu_{-1} + \Lambda^4(W) \otimes \mu_{-1} \quad (32)$$

$$L^2 = W^* + W \quad (33)$$

$$\begin{aligned} L^3 &= \Lambda^4(W^*) \otimes \mu_1 + \Lambda^2(W^*) \otimes \mu_1 + \mu_1 \cong \\ &\cong W \otimes \mu_{-1} + \Lambda^3(W) \otimes \mu_{-1} + \mu_1 \end{aligned} \quad (34)$$

$$L^4 = \Lambda^2(W) + \Lambda^2(W^*) + W \otimes W^* \quad (35)$$

$$\dots \quad (36)$$

The above formulas are written in such a way that the first summand in every line is a representation of P ; the same is true for the sum of first two summands.

2.4 Euler characteristics

The statements of Section 2.2 permit us to calculate the Euler characteristics of $H^\bullet(L, N)$ and $H_\bullet(L, N)$. Recall that for every complex (=differential graded abelian group) $C = \sum C^k$ we can define cohomology $H = \sum H^k$ and Euler characteristic $\chi = \chi(H) = \sum (-1)^k \dim H^k$. If C has only finite number of

graded components (C^k does not vanish only for finite number of k) we can represent χ in the form

$$\chi(H) = \sum (-1)^k \alpha^k \quad (37)$$

where $\alpha^k = \dim C^k$. It is important to notice that χ can be expressed in terms of α^k also in the case when the number of graded components of C is infinite. (We assume that the number of non-vanishing cohomology groups is finite.) Namely, it is easy to check that for appropriate choice of factors $\rho_\epsilon(k)$ we have

$$\chi = \lim_{\epsilon \rightarrow 0} \sum \rho_\epsilon(k) (\alpha^{2k} - \alpha^{2k-1}). \quad (38)$$

Here $\rho_\epsilon(k) \rightarrow 1$ as $\epsilon \rightarrow 0$ and it is a fast decreasing function of k as $|k| \rightarrow \infty$. (For example, if α^k grows as k^n we can take $\rho_\epsilon(k) = 1 + \frac{\epsilon}{|k|^m}$ where $m > n + 2$. For exponential growth $\alpha^k \sim e^{sk}$, which is more appropriate in our setting we take $\rho_\epsilon(k) = 1 + \epsilon e^{-(s+1)k}$). If a Lie algebra \mathfrak{g} acts on C (more precisely, if C is a differential \mathfrak{g} -module) this Lie algebra acts also on cohomology and we can define χ as an element of the representation ring of \mathfrak{g} .⁷ All above statements remain correct in this more general situation after appropriate modifications (α^k should be considered as the class of C^k in the representation ring).

We will be interested in Euler characteristic in the case when the group $\text{Spin}(10)$ acts on L -module N . (More precisely, we assume that N is a module with respect to semidirect product of $\text{spin}(10)$ and L .) Then we can consider Euler characteristics of $H^\bullet(L, N)$ and of $H_\bullet(L, N)$ -modules as virtual $\text{spin}(10)$ -modules and express them in terms of graded components of N , \mathcal{S} and \mathcal{S}^* considered as $\text{spin}(10)$ -modules. This calculation will be given in Part II.

Suppose that an algebra A is equipped with an action of a compact group G , which acts by automorphisms of A . We can define Hilbert series of A with values in G characters (or, equivalently, with values in the representation ring of G). Let $T_i(g)$ be an operator that acts in the i -th graded component of A .

⁷Recall that elements of representation ring are virtual representations; we define χ as virtual representation obtained as an alternating sum of representations H^k . For Lie algebra of compact group instead of representation ring one can talk about the ring of characters.

We define a formal power series $A(t, g)$ by the formula

$$A(t, g) = \sum_{i \geq 0} \text{tr}(T_i(g)) t^i$$

Proposition 9 *Let A be a Koszul algebra, equipped with G -action. Then the group G also acts on $A^!$ and there is an equality*

$$A(t, g)A^!(-t, g^{-1}) = 1$$

Proof. It is a trivial adaptation of the proof [23] for the case of algebra with G -action. Obviously the space of relations of $A^!$ is invariant with respect to the G -action. The complex $A \otimes (A^!)^*$ has trivial cohomology by the definition of Koszul algebra. It decomposes into a direct sum of acyclic complexes $K_n = \bigoplus_{i+j=n} A_i \otimes (A_j^!)^*$. The generating function of Euler characteristics of K_n is equal to the constant function 1. But it also equals to the product of the generating functions $A(t, g)A^!(-t, g^{-1})$. (We use here the fact that the character of the dual representation is expressed in terms of the character $\rho(g)$ of original representation as $\rho(g^{-1})$.) ■

The structure of \mathcal{S} as $\text{spin}(10)$ -module was described in [15] by means of Borel-Weyl-Bott theorem; it is easy to check that

$$\mathcal{S}(g, t) = (1 - V(g)t^2 - S(g)t^3 + S(g^{-1})t^5 + V(g^{-1})t^6 - t^8)\text{Sym}(S)(t, g).$$

where S stands for spinor representation and V for vector representation. One can use this statement to calculate the character. (One can also calculate the character directly as in [21].) The information about \mathcal{S} permits us to analyze the structure of Koszul dual algebra $U(L)$.

Corollary 10 *The Hilbert series $U(L)(t, g)$ of the universal enveloping $U(L)$ is equal to*

$$\frac{\Lambda(S)(t, g)}{1 - V(g^{-1})t^2 + S(g^{-1})t^3 - S(g)t^5 + V(g)t^6 - t^8}$$

Proof. Apply Proposition 9 to algebras \mathcal{S} , $\mathcal{S}^! \cong U(L)$ and $\text{Sym}(S)^! \cong \Lambda(S^*)$.

■

3 Infinitesimal SUSY Deformations

Let us consider an infinitesimal deformation $\delta\mathcal{L}$ of a Lagrangian \mathcal{L} . If an infinitesimal deformation $\delta'\mathcal{L}$ is obtained from $\delta\mathcal{L}$ by means of a field redefinition then $\delta'\mathcal{L} = \delta\mathcal{L} + \text{total derivative}$ ⁸ on the solutions of EM for \mathcal{L} . (This means that action functionals corresponding to infinitesimal deformations $\delta\mathcal{L}, \delta'\mathcal{L}$ coincide on solutions of EM for \mathcal{L} .)

The converse statement is also true. Therefore we will identify infinitesimal deformations $\delta\mathcal{L}$ and $\delta'\mathcal{L}$ of \mathcal{L}_{SYM} if $\delta'\mathcal{L} = \delta\mathcal{L} + \text{total derivative}$ on the solutions of EM for \mathcal{L}_{SYM} .⁹

We will be interested in deformations of SYM that are defined simultaneously for all gauge groups $U(N)$. Let us consider first the Lagrangian \mathcal{L}_{SYM} reduced to a point. The deformation of the kind we are interested in are single-trace deformations: they can be represented in the form $\text{tr}\Lambda$, where Λ is an arbitrary non-commutative polynomial in terms of the fields of the reduced

⁸We say that a function on \mathbb{R}^n is a total derivative if it can be represented in the form $\frac{\partial}{\partial x^i} H^i$. In more invariant way one can say that the differential form of degree n corresponding to this function should be exact.

⁹To reach a better understanding of the above statements we will discuss a finite-dimensional analogy.

Any function

$$f : \mathbb{C}^n \rightarrow \mathbb{C} \tag{39}$$

can be deformed by adding an arbitrary function g multiplied by an infinitesimal parameter. It is not true however that the space of deformations of f coincides with the space $\hat{\mathcal{O}}$ of all g . The reason is that there are trivial deformations of f obtained by a change of parametrization of \mathbb{C}^n . A vector field ξ on \mathbb{C}^n defines an infinitesimal change of coordinates, under which f transforms to $f_\xi = \xi^i \frac{\partial f}{\partial x^i}$. The space $Vect f$ of functions f_ξ forms a subspace of trivial infinitesimal deformations. The quotient $\hat{\mathcal{O}}/Vect f$ is the formal tangent space to the space of nontrivial deformations of (39).

Under some conditions of regularity one can identify $\hat{\mathcal{O}}/Vect f$ with the algebra of functions on the set of critical points of f . (If this set is considered as a scheme the conditions of regularity are not necessary.)

In field theories this identification corresponds to identification of off-shell classes of infinitesimal deformations of an action functional with action functionals considered on shell (on the solutions of EM).

theory. (The fields form an array of $N \times N$ skew-hermitian matrix variables $A_1, \dots, A_n, \chi^1, \dots, \chi^{16}$ of suitable parity in the theory with the gauge group $U(N)$. Reality conditions are left out of scope of our analysis, and we simply let the fields to be elements of complex matrices Mat_N . We are working with all of these groups simultaneously, hence we consider the fields as formal non-commuting variables, i.e. as generators of free graded associative algebra). The (super)trace of (super)commutator vanishes, hence we can identify the space of deformations with $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$ where \mathcal{A} stands for the free graded associative algebra, generated by symbols of fields

$$D_i, \chi^\alpha. \quad (40)$$

¹⁰ However, we should take into account that the deformations can be equivalent (related by a change of variables). As we have seen this means that the action functionals coincide on shell. This means that the space of equivalence classes of deformations can be identified with $U(YM)/[U(YM), U(YM)]$ where $U(YM)$ can be interpreted as an associative algebra generated by the fields of SYM theory reduced to a point with relations coming from the equations of motion (see Section 2 for more detail). ¹¹

Similar results are true for non-reduced SYM theory. In this case we consider deformations of the form $\text{tr} \Lambda$, where Λ is a gauge covariant local expression or, in other words an element of $U(TYM)$ (see Section 1).

We are saying that infinitesimal deformation $\delta \mathcal{L}$ is supersymmetric if $\theta_\alpha \delta \mathcal{L}$ is trivial deformation, i.e. it can be represented as a total derivative

$$\theta_\alpha \delta \mathcal{L} = \frac{\partial}{\partial x^i} H^i \quad (41)$$

on equations of motion of \mathcal{L}_{SYM} . Poincaré invariance is defined in a similar way.

¹⁰ We can identify this space also with the space of cyclic words in the alphabet where letters correspond to the fields.

¹¹ This space has also interpretation in terms of Hochschild homology $HH_0(U(YM), U(YM))$ or in terms of cyclic homology.

There exist an infinite number of infinitesimal super Poincaré invariant deformations. Most of them are given by a simple general formula below, but there are three exceptional deformations which do not fit into this formula. The first was discussed earlier in [1].

$$\begin{aligned}
\delta\mathcal{L}_{16}(\nabla, \chi) = & \text{tr} \left(\frac{1}{8} F_{mn} F_{nr} F_{rs} F_{sm} - \frac{1}{32} (F_{mn} F_{mn})^2 \right. \\
& + i \frac{1}{4} \chi^\alpha \Gamma_{m\alpha\beta} (\nabla_n \chi^\beta) F_{mr} F_{rn} \\
& - i \frac{1}{8} \chi^\alpha \Gamma_{mnr\alpha\beta} (\nabla_s \chi^\beta) F^{mn} F_{rs} \\
& + \frac{1}{8} \chi^\alpha \Gamma_{\alpha\beta}^m (\nabla_n \chi^\beta) \chi^\gamma \Gamma_{m\gamma\delta} (\nabla_n \chi^\delta) \\
& \left. - \frac{1}{4} \chi^\alpha \Gamma_{\alpha\beta}^m (\nabla_n \chi^\beta) \chi^\gamma \Gamma_{n\gamma\delta} (\nabla_m \chi^\delta) \right)
\end{aligned} \tag{42}$$

It is convenient to introduce a grading on space of fields (2). We suppose that grading is multiplicative and $\deg(\nabla_i) = 2$, $\deg(\chi^\alpha) = 3$. This grading is related to the grading with respect to α' , that comes from string theory, by the formula

$$\deg_{\alpha'} = \frac{\deg - 8}{4}. \tag{43}$$

Subscript in $\delta\mathcal{L}_{16}$ in the formula (42) stands for the grading of infinitesimal Lagrangian. Lagrangian $\delta\mathcal{L}_{16}$ is a super Poincaré invariant deformation of lowest possible degree.¹² The next linearly independent infinitesimal super Poincaré invariant deformation (of degree 20) was found in [7]. It has the following Lagrangian

$$\begin{aligned}
\delta\mathcal{L}_{20}(\nabla, \chi) = & f^{XYZ} f^{VWZ} \left[2 F_{ab}^X F_{cd}^W \nabla_e F_{bc}^V \nabla_e F_{ad}^Y - 2 F_{ab}^X F_{ac}^W \nabla_d F_{be}^V \nabla_d F_{ce}^Y \right. \\
& + F_{ab}^X F_{cd}^W \nabla_e F_{ab}^V \nabla_e F_{cd}^Y \\
& - 4 F_{ab}^W \nabla_e F_{bd}^Y \chi^{\alpha X} \Gamma_{\alpha\beta a} \nabla_d \nabla_c \chi^{\beta V} - 4 F_{ab}^W \nabla_e F_{bd}^Y \chi^{\alpha X} \Gamma_{\alpha\beta d} \nabla_a \nabla_c \chi^{\beta V} \\
& \left. + 2 F_{ab}^W \nabla_e F_{de}^Y \chi^{\alpha X} \Gamma_{\alpha\beta ade} \nabla_b \nabla_c \chi^{\beta V} + 2 F_{ab}^W \nabla_e F_{de}^Y \chi^{\alpha X} \Gamma_{\alpha\beta abd} \nabla_e \nabla_c \chi^{\beta V} \right] +
\end{aligned}$$

¹²We will treat the truly lowest order deformation $\delta\mathcal{L} = \mathcal{L}_{SYM}$ as trivial.

$$\begin{aligned}
& + f^{XYZ} f^{UVW} f^{TUX} \left[4 F_{ab}^Y F_{cd}^Z F_{ac}^V F_{be}^W F_{de}^T + 2 F_{ab}^Y F_{cd}^Z F_{ab}^V F_{ce}^W F_{de}^T \right. \\
& - 11 F_{ab}^Y F_{cd}^Z F_{cd}^V \chi^{\alpha T} \Gamma_{\alpha\beta a} \nabla_b \chi^{\beta W} + 22 F_{ab}^Y F_{cd}^Z F_{ac}^V \chi^{\alpha T} \Gamma_{\alpha\beta b} \nabla_d \chi^{\beta W} \\
& + 18 F_{ab}^Y F_{cd}^V F_{ac}^W \chi^{\alpha T} \Gamma_{\alpha\beta b} \nabla_d \chi^{\beta Z} + 12 F_{ab}^T F_{cd}^Y F_{ac}^V \chi^{\alpha Z} \Gamma_{\alpha\beta b} \nabla_d \chi^{\beta W} \\
& + 28 F_{ab}^T F_{cd}^Y F_{ac}^V \chi^{\alpha W} \Gamma_{\alpha\beta b} \nabla_d \chi^{\beta Z} - 24 F_{ab}^Y F_{cd}^V F_{ac}^T \chi^{\alpha W} \Gamma_{\alpha\beta b} \nabla_d \chi^{\beta Z} \\
& + 8 F_{ab}^T F_{cd}^Y F_{ac}^Z \chi^{\alpha V} \Gamma_{\alpha\beta b} \nabla_d \chi^{\beta W} - 12 F_{ab}^T F_{ac}^Y \nabla_b F_{cd}^V \chi^{\alpha Z} \Gamma_{\alpha\beta d} \chi^{\beta W} \\
& - 8 F_{ab}^Y F_{ac}^T \nabla_b F_{cd}^V \chi^{\alpha Z} \Gamma_{\alpha\beta d} \chi^{\beta W} + 22 F_{ab}^V F_{ac}^Y \nabla_b F_{cd}^T \chi^{\alpha Z} \Gamma_{\alpha\beta d} \chi^{\beta W} \\
& - 4 F_{ab}^Y F_{cd}^T \nabla_e F_{ac}^V \chi^{\alpha Z} \Gamma_{\alpha\beta bde} \chi^{\beta W} + 4 F_{ab}^Y F_{ac}^T \nabla_c F_{de}^V \chi^{\alpha Z} \Gamma_{\alpha\beta bde} \chi^{\beta W} \\
& + 4 F_{ab}^T F_{cd}^Y F_{ce}^V \chi^{\alpha Z} \Gamma_{\alpha\beta abd} \nabla_e \chi^{\beta W} - 8 F_{ab}^Y F_{cd}^T F_{ce}^V \chi^{\alpha Z} \Gamma_{\alpha\beta abd} \nabla_e \chi^{\beta W} \\
& + 6 F_{ab}^V F_{cd}^Y F_{ce}^W \chi^{\alpha Z} \Gamma_{\alpha\beta abd} \nabla_e \chi^{\beta T} + 5 F_{ab}^V F_{cd}^W F_{ce}^Y \chi^{\alpha Z} \Gamma_{\alpha\beta abd} \nabla_e \chi^{\beta T} \\
& + 6 F_{ab}^Y F_{ac}^T F_{de}^V \chi^{\alpha Z} \Gamma_{\alpha\beta bcd} \nabla_e \chi^{\beta W} - 2 F_{ab}^Y F_{ac}^T F_{de}^Z \chi^{\alpha V} \Gamma_{\alpha\beta bcd} \nabla_e \chi^{\beta W} \\
& + 4 F_{ab}^Y F_{ac}^V F_{de}^Z \chi^{\alpha W} \Gamma_{\alpha\beta bcd} \nabla_e \chi^{\beta T} + 4 F_{ab}^T F_{cd}^V F_{ce}^Y \chi^{\alpha Z} \Gamma_{\alpha\beta abd} \nabla_e \chi^{\beta W} \\
& - 4 F_{ab}^Y F_{cd}^V F_{ce}^W \chi^{\alpha Z} \Gamma_{\alpha\beta abd} \nabla_e \chi^{\beta T} \\
& \left. + \frac{1}{2} F_{ab}^Y F_{cd}^T F_{ef}^V \chi^{\alpha Z} \Gamma_{\alpha\beta abcde} \nabla_f \chi^{\beta W} + \frac{1}{2} F_{ab}^Y F_{cd}^T f_{ef}^Z \chi^{\alpha V} \Gamma_{\alpha\beta abcde} \nabla_f \chi^{\beta W} \right].
\end{aligned} \tag{44}$$

In these formulas capital Roman are Lie algebra indices, f^{XYZ} are structure constants of the gauge group Lie algebra.

The way to get the formula (44) will be described below.

One can construct a SUSY-invariant deformation by the formula:

$$\delta\mathcal{L} = \text{Atr}G \tag{45}$$

where the operator A is given by

$$A = \theta_1 \dots \theta_{16}. \tag{46}$$

Here $\text{tr}G$ is a gauge invariant expression (we can consider G as an element of $U(YM)$). If G is $\text{Spin}(10)$ -invariant the deformation $\delta\mathcal{L} = \text{Atr}G$ is super Poincaré invariant.

Let us check that infinitesimal deformation (45) is supersymmetric. It is sufficient to prove that $\theta_\alpha \delta\mathcal{L}$ is a total derivative, i.e. $\theta_\alpha \delta\mathcal{L} = \frac{\partial}{\partial x^i} H^i$ on shell (on the equations of motion of \mathcal{L}_{SYM}). To prove this fact we notice that the

anti-commutator $[\theta_\alpha, \theta_\beta]$ is a total covariant derivative as follows from (5). It follows from the same formula that θ_α^2 is a total covariant derivative. Calculating $\theta_\alpha \text{Atr}G$ we are moving θ_α using (5) until we reach θ with the same index. Then we use a formula for θ_α^2 :

$$\begin{aligned} \theta_\alpha \delta \mathcal{L} &= \text{tr}(\theta_\alpha \theta_1 \cdots \theta_{16} G) = \\ &= \sum_{\gamma=1}^{\alpha-1} (-1)^\gamma \Gamma_{\alpha\gamma}^k \text{tr}(\theta_1 \cdots \theta_{\gamma-1} D_k \theta_{\gamma+1} \cdots \theta_{16} G) + \\ &+ \frac{1}{2} \sum_{\gamma=1}^{\alpha-1} (-1)^\alpha \Gamma_{\alpha\alpha}^k \text{tr}(\theta_1 \cdots \theta_{\alpha-1} D_k \theta_{\alpha+1} \cdots \theta_{16} G). \end{aligned} \quad (47)$$

Expressions $\text{tr}(D_k \theta_\alpha \cdots \theta_{16} G) = \frac{\partial}{\partial x_k} \text{tr}(\theta_\alpha \cdots \theta_{16} G)$ are total derivatives. Expressions $\text{tr}(\theta_1 \cdots \theta_{\gamma-1} D_k \theta_{\gamma+1} \cdots \theta_{16} G)$ are multiple supersymmetry transformations of total derivatives. Hence due to equation (6) $\text{Atr}G$ is also a total derivative on the equations of motion for \mathcal{L}_{SYM} .

The reader will recognize in (46) a 10-dimensional analog of θ -integration in theories admitting superspace formulation with manifest supersymmetries.

The formula (45) is fairly general and works also for reduced theories. In particular the above considerations can be used to describe all infinitesimal deformations of YM theory reduced to a point. Namely we have the following theorem.

Theorem 11 *Every infinitesimal super Poincaré-invariant deformation of \mathcal{L}_{SYM} reduced to a point is a linear combination of \mathcal{L}_{16} and a deformation having a form $\text{Atr}(G)$, where G is an arbitrary $\text{Spin}(10)$ -invariant combination of products of A_i and χ^α .*

To formulate the corresponding statement in case of unreduced \mathcal{L}_{SYM} we should generalize the above consideration a little bit. We notice that infinitesimal deformation of \mathcal{L}_{SYM} reduced to a point can be lifted to a deformation of unreduced \mathcal{L}_{SYM} if it has the form $\text{tr}\Lambda$ where Λ is a gauge covariant local expression up to commutator terms that disappear under the sign of trace. The rule of turning a function of matrices A_i, χ^α into a Lagrangian on a space of

connections and spinors is simple. We formally replace A_i by ∇_i and let χ to be x -dependent. By definition an infinitesimal deformation of the reduced theory is local if the above procedure succeeds and defines a Lagrangian. It is not hard to see that the Lagrangian is defined unambiguously. It is not true that all Lagrangians in the reduced theory give rise to a Lagrangian in ten dimensional theory. For example an expression $\text{tr} A_i A_i$ would define a Lagrangian of reduced theory, but

$$\Delta = \text{tr} \nabla_i \nabla_i \quad (48)$$

does not make sense as a ten-dimensional Lagrangian.

Of course, if $\text{tr} G$ itself is a gauge-covariant local expression, the expression $\text{Atr} G$ is also local. However there are situations when $\text{tr} G$ is not of this kind but still $\text{Atr}(G)$ specifies a gauge-invariant local expression; then this expression can be considered as a Lagrangian of SUSY deformation of unreduced \mathcal{L}_{SYM} .

Gauge covariant local expression can be considered as elements of the algebra $U(TYM)$ defined in Section 2.1. This means that infinitesimal deformations of ten dimensional SYM theory can be identified with elements of $U(TYM)/[U(TYM), U(TYM)]$.

Our homological computations [20] show that the number of linearly independent Poincaré invariant deformations in ten dimensional theory which do not have the form $\text{Atr}(G)$ where $G \in U(TYM)$, but can be written in this form with $G \in U(YM)$ is equal to two.

To construct the first one we take G to be the "Laplacian" (48).

We have

$$\text{Atr}(\Delta) = 2\text{tr}((A\nabla_i)\nabla_i) + \dots, \quad (49)$$

where the dots represent gauge-invariant local terms. This follows from formula (3). It remains to prove that $\text{tr}((A\nabla_i)\nabla_i)$ is equivalent to a local expression (recall that we identify deformations related by field redefinition). It follows from the remarks at the beginning of the section that instead of working with $\text{tr}((A\nabla_i)\nabla_i)$ we can work with $(A\mathbf{D}_i)\mathbf{D}_i$ considered as an element of

$U(YM)/[U(YM), U(YM)]$.

The commutators with θ_α act as supersymmetries on D_i, χ^β .

In the algebra $U(L)$ we can represent $A(D_i)$ as the multiple commutator:

$$A(D_i) = [\theta_1, \dots, [\theta_{16}, D_i] \dots].$$

We have more then four θ 's in a row applied to θ_β . We see that $A(D_i)$ is a commutator

$$A(D_i) = [D_k, \psi_{ki}] + [\chi^\alpha, \psi_{\alpha i}] \quad (50)$$

in $U(YM)$, where $\psi_{ki}, \psi_{\alpha i}$ are gauge-covariant local expressions. We have the following line of identities where we can neglect commutator terms:

$$A(D_i)D_i = [D_k, \psi_{ki}]D_i + [\chi^\alpha, \psi_{\alpha i}]D_i = -\psi_{ki}[D_k D_i] + [D_k, \psi_{ki}D_i] - \psi_{\alpha i}[D_i, \chi^\alpha] + [\chi^\alpha, \psi_{\alpha i}D_i].$$

We obtain that $\text{tr}(A(D_i)D_i) = \text{tr}(\psi_{ki}[D_k D_i]) - \text{tr}(\psi_{\alpha i}[D_i, \chi^\alpha])$.

One can check that supersymmetric deformation obtained from $\Delta = G_1$ is equivalent to (44).

One can prove that similar considerations can be applied to

$$G_2 = a \text{tr}(F_{i_2 i_3} F_{i_2 i_3} D_{i_1} D_{i_1}) + b \text{tr}(\Gamma_{\alpha\beta}^{i_2} [D_{i_1}, \chi^\alpha] \chi^\beta D_{i_2} D_{i_1}) + c \text{tr}(\Gamma_{\alpha\beta}^{i_1 i_2 i_3} F_{i_2 i_3} \chi^\alpha \chi^\beta D_{i_1})$$

for an appropriate choice of constants a, b, c . Corresponding deformation will be denoted by $\delta\mathcal{L}_{28}$.

To give a more conceptual proof that the deformation $\delta\mathcal{L}_{20}$ corresponding to $\Delta = D_i D_i$ and the deformation $\delta\mathcal{L}_{28}$ are deformations of ten dimensional SYM theory we also will work in the algebra $U(L)$. Let ad_{θ^α} be an operator acting on the space $U(L)$ and defined by the formula

$$ad_{\theta^\alpha}(x) = [\theta_\alpha, x]$$

. Let $P(\theta_1, \dots, \theta_{16})$ be an arbitrary non-commutative polynomial in θ_α . We define ad_P as $P(ad_{\theta_1}, \dots, ad_{\theta_{16}})$. In these notations the operator A coincides with $ad_{\theta_1 \dots \theta_{16}}$.

One can prove the following general statement:

Let us take a collection $(\mathbf{G}^\alpha), \alpha = 1, \dots, 16$ of odd elements of $U(TYM)$ that satisfies

$$[\theta_\alpha, \mathbf{G}^\alpha] = 0. \quad (51)$$

Let us define \mathbf{G} by the formula

$$\mathbf{G} = \theta_\alpha \mathbf{G}^\alpha.$$

Then (up to commutator terms) $A\mathbf{G} \in U(TYM)$ and therefore specifies a deformation of ten-dimensional theory. It follows from our previous consideration that this deformation is supersymmetric.

Let us introduce an operator A^α which is equal to $ad_{\theta_1 \dots \hat{\theta}_\alpha \dots \theta_{16}}$. As usual \wedge stands for omission. The equation (51) implies that

$$\begin{aligned} 0 &= A^\beta \left(\sum_\alpha [\theta_\alpha, \mathbf{G}^\alpha] \right) = \sum_{\alpha \neq \beta} A^\beta [\theta_\alpha, \mathbf{G}^\alpha] + A^\beta [\theta_\beta, \mathbf{G}^\beta] = \\ &= \Gamma_{\alpha\alpha}^i ad_{\theta_1 \dots \hat{\theta}_\beta \dots \hat{\theta}_\alpha D_i \dots \theta_{16}} \mathbf{G}^\alpha + ad_{\theta_1 \dots \hat{\theta}_\beta \dots \theta_\alpha [\theta_{\alpha+1} \dots \theta_{16}, \theta_\alpha]} \mathbf{G}^\alpha + \\ &+ (-1)^{16-\beta} A\mathbf{G}^\beta + ad_{\theta_1 \dots \hat{\theta}_\beta [\theta_{\beta+1} \dots \theta_{16}, \theta_\beta]} = \\ &= x^\beta + y^\beta + z^\beta + w^\beta \end{aligned} \quad (52)$$

The commutation relations (13), (16) imply that the terms x^β, y^β and w^β satisfy

$$\begin{aligned} x^\beta &= [D_i, X_i^\beta], \\ y^\beta &= [D_i, Y_i^\beta], \\ w^\beta &= [D_i, W_i^\beta] \end{aligned}$$

for some $X_i^\beta, Y_i^\beta, W_i^\beta$ from $U(TYM)$. $A\mathbf{G}^\beta$ coincides with $(-1)^{15+\beta} [D_i, (X_i^\beta + Y_i^\beta + W_i^\beta)]$, which we denote by $[D_i, U_i^\beta]$. But the term $\theta_\beta A\mathbf{G}^\beta$ is equal to $\theta_\beta [D_i, U_i^\beta]$.

Neglecting commutator terms we get

$$\theta_\beta A\mathbf{G}^\beta = [D_i, \theta_\beta] U_i^\beta = \Gamma_{i\beta\alpha} \chi^\alpha U_i^\beta \in U(TYM). \quad (53)$$

From this we infer that $A\theta_\alpha \mathbf{G}^\alpha = \sum_\alpha \frac{16!}{\alpha!(16-\alpha)!} ad_{\theta_1 \dots \theta_\beta} (\theta_\alpha) ad_{\theta_{\beta+1} \dots \theta_{16}} (\mathbf{G}^\alpha)$. It follows from commutation relations (13) and (16) that all terms besides one

belong to $U(TYM)$. The only term for which this is not evident is $\theta_\alpha A G^\alpha$, but the formula (53) takes care of it.

As we have mentioned already we can take $G^\alpha = \chi^\alpha$. Then

$$G = \theta_\alpha \chi^\alpha.$$

Using formulas (13) and (16) we obtain that up to a constant factor and up to commutator terms G coincides with $D_i D_i$.

We obtain that $\delta \mathcal{L}_{20}$ is a super- Poincaré invariant deformation of ten dimensional SYM theory. The construction of elements G^α giving the deformation $\delta \mathcal{L}_{28}$ is much more involved; the general method that allows to solve (51) will be described in Part II.

Theorem 12 *Every infinitesimal super-Poincaré invariant deformation of Lagrangian \mathcal{L}_{SYM} is a linear combination of $\delta \mathcal{L}_{16}$ given by the formula (42), $\delta \mathcal{L}_{20} = \text{Atr}(G_1)$, $\delta \mathcal{L}_{28} = \text{Atr}(G_2)$ and a deformation of a form $\text{Atr}(G)$ where G is an arbitrary Poincaré-invariant combination of products of covariant derivatives of curvature F_{ij} and spinors χ^α .*

There is a finer decomposition of the linear space of equivalence classes of Lagrangians. Any Lagrangian \mathcal{L} under consideration has the form $\mathcal{L} = \text{tr} Y(\nabla, \chi)$, where Y is some non-commutative polynomial in ∇_i and χ^α . Let the non-commutative polynomial Y be a linear combination of commutators. Then of course $\text{tr} Y \equiv 0$, however if Y, Y' are commutators then

$$\text{tr} Y Y' \tag{54}$$

could be nonzero. The grading $\text{deg}_{[\]}$ of a Lagrangian of the form $\text{tr} Y Y'$ by definition is equal to two (to the number of commutators in the product under the trace in (54)).¹³ For example the basic Lagrangian \mathcal{L}_{SYM} has degree $\text{deg}_{[\]}$ equal to two. Likewise we can define Lagrangians of arbitrary degree $\text{deg}_{[\]}$.

¹³Lagrangians of this kind make sense not only for the gauge group $U(N)$, but also for an arbitrary compact gauge group G because they can be written intrinsically in terms of the commutator and the invariant inner product of the Lie algebra of G .

The equations of motion of YM theory are compatible with classification of Lagrangians by $\deg_{[\]}$ in the sense that Lagrangians of different degree are not equivalent.

The following table is a result of classification of linearly independent on-shell supersymmetric Lagrangians of low degree. The numbers in the body of the table represent dimensions of spaces of super Poincaré invariant deformations of degrees $(\deg_{[\]}, \deg_{\alpha'})$.

	1	2	3	4	5	6	7	8	$k = \deg_{\alpha'}$
2			1		1	3	18	172	...
3							13	281	...
4		1			1	2	20	267	...
5							1	68	...
6							1	17	...
7									...
$p = \deg_{[\]}$

(55)

The entry in the second column corresponds to the Lagrangian (42), the entry in the third column corresponds to the Lagrangian (44).

4 Homological Approach to Infinitesimal Deformations

In this section we will describe a reduction of the problem of infinitesimal SUSY deformations of SYM to a homological problem. A general way to give homological formulation of a problem of classification of deformations will be described in Section 6 and in Appendix A; the relation of this way to the approach of present section will be studied in Appendix C.

First of all we consider infinitesimal deformations of SYM reduced to a point. As we have seen, this theory can be expressed in terms of algebra $U(YM)$. We will regard the deformations of this theory as deformations of algebra $U(YM)$. In other words we think about deformation as of family of multiplications on linear space $U(YM)$ depending smoothly on parameter α' . In the case of infinitesimal deformations we assume that $\alpha'^2 = 0$ (i.e. we neglect higher order

terms with respect to α'). We say that deformation is supersymmetric if it is possible to deform the SUSY algebra action on $U(YM)$ in such a way that it consists of derivations of the deformed multiplication.

Theorem 13 *Every cohomology class $\lambda \in H^2(L, U(YM)) = H^2(L, \text{Sym}(YM))$ specifies an infinitesimal supersymmetric deformation of $U(YM)$.*

We consider here $U(YM)$ as a representation of Lie algebra L . Due to Poincaré-Birkhoff-Witt theorem this representation is isomorphic to $\text{Sym}(YM)$.

We will start with general statement about deformations of associative algebra A . The multiplication in this algebra can be considered as a bilinear map $m : A \otimes A \rightarrow A$. An infinitesimal deformation $m + \delta m$ of this map specifies an associative multiplication if

$$\delta m(a, b)c + \delta m(ab, c) = a\delta m(b, c) + \delta m(a, bc).$$

This condition means that δm is a two-dimensional Hochschild cocycle with coefficients in A (see Appendix A). Identifying equivalent deformations we obtain that infinitesimal deformations of associative algebra are labeled by the elements of Hochschild cohomology $HH^2(A, A)$. (Two deformations are equivalent if they are related by linear transformation of A .)

Applying this statement to the algebra $U(YM)$ we obtain that the infinitesimal deformations of this algebra are labeled by the elements of $HH^2(U(YM), U(YM))$.

Let us consider now the Hochschild cohomology $HH^2(U(L), U(YM))$. (Notice that $U(YM)$ is an ideal in $U(L)$, hence it can be regarded as a $U(L)$ -bimodule.) We can consider the natural restriction map $HH^2(U(L), U(YM)) \rightarrow HH^2(U(YM), U(YM))$; we will check that the image of this map consists of supersymmetric deformations. Let us notice first of all that $L = L^1 + YM$ and the derivations γ_a corresponding to the elements $a \in L^1$ act on YM as supersymmetries; this action can be extended to $U(YM)$ and specifies an action on Hochschild cohomology, in particular, on the space of deformations $HH^2(U(YM), U(YM))$. (The derivation γ_a is defined by the formula $\gamma_a(x) = [a, x]$.) On L one can consider γ_a as an inner derivation, hence its action on the

cohomology $HH^2(U(L), U(YM))$ is trivial. (This follows from well known results, see, for example, [14].) This means that supersymmetry transformations act trivially on the image of $HH^2(U(L), U(YM))$ in $HH^2(U(YM), U(YM))$ (in the space of deformations).

To obtain the statement of the theorem it is sufficient to notice that the Hochschild cohomology of the enveloping algebra of Lie algebra can be expressed in terms of Lie algebra cohomology (see (24), Section 2.2),

Theorem (13) gives a homological description of supersymmetric deformations of the equations of motion. We can use homological methods to answer the question: when the deformed EM come from a Lagrangian. As we have seen in Section 3 the space of infinitesimal Lagrangian deformations of SYM theory reduced to a point can be identified with $U(YM)/[U(YM), U(YM)] = HH_0(U(YM), U(YM))$. Lagrangian deformation generates a deformation of EM, hence there exists a map $U(YM)/[U(YM), U(YM)] \rightarrow HH^2(U(YM), U(YM)) = H^2(YM, U(YM))$. It turns out (see [16] and [20]) that the image of this map has a finite codimension in $H^2(YM, U(YM))$ and it is onto for $\text{Spin}(10)$ -invariant elements. This means that all Poincaré invariant infinitesimal deformations of EM are Lagrangian deformations.

Let us consider now deformations of supersymmetric deformations of supersymmetric YM theory in ten-dimensional case (SYM theory). The description of these deformations is similar to reduced case.

Theorem 14 *Every element $\lambda \in H^2(L, U(TYM))$ specifies a supersymmetric deformation of SUSY YM.*

In the proof we interpret the deformations of SYM theory as deformations of the algebra $U(TYM)$ and identify infinitesimal deformations with elements of Hochschild homology $HH_0(U(TYM), U(TYM))$. However, the proof is more complicated; it is based on results of Section 6 and Appendix C. It is shown in Appendix C that the elements of higher cohomology groups also correspond to supersymmetric deformations, however only elements of H^2 give non-trivial super-Poincaré invariant infinitesimal deformations of equations of motion.

Notice that using the results of the Section 2.4 one can calculate Euler characteristics of groups $H^k(L, U(YM))$, $H^k(L, U(TYM))$ and corresponding homology groups considered as $\text{Spin}(10)$ -modules. This calculation will be performed in more general situation in Part II of present paper.

5 Calculation of Cohomology

The calculation will be based on Corollaries 5,6 (Section 2.2).

We mentioned in Section 2.1 that the algebra \mathcal{S} is related to the manifold of pure spinors $C\mathcal{Q}$ and to the corresponding compact manifold \mathcal{Q} . Namely \mathcal{S}_k can be interpreted as a space of holomorphic sections of line bundle $\mathcal{O}(k)$ over \mathcal{Q} . In other words

$$\mathcal{S}_k = H^0(\mathcal{Q}, \mathcal{O}(k)) \text{ for } k \geq 0. \quad (56)$$

One can prove [19] that all other cohomology groups $H^i(\mathcal{Q}, \mathcal{O}(k))$ of \mathcal{Q} with coefficients in line bundles $\mathcal{O}(k)$ are zero except

$$H^{10}(\mathcal{Q}, \mathcal{O}(k)) = \mathcal{S}_{-k-8}^* \text{ for } k \leq -8. \quad (57)$$

The proof is based on Borel-Weil-Bott theorem.¹⁴

¹⁴ Borel-Weil-Bott theory deals with calculation of the cohomology of G/P with coefficients in G -invariant holomorphic vector bundles over G/P . Here G/P is a compact homogeneous space, P is a complex subgroup of complex Lie group G . These bundles correspond to complex representations of the subgroup P ; more precisely, the total space of vector bundle \mathcal{E} corresponding to P -module E (to a representation of P in the space E) can be obtained from $E \times G$ by means of factorization with respect to the action of P .

Usually Borel-Weil-Bott theorem is applied in the case when the representation of P is one-dimensional (in the case of line bundles); it describes the cohomology as a representation of the group G . However, more general case also can be treated [5].

We suppose that the group G is connected and the homogeneous space G/P is simply connected; then G/P can be represented as $M/P \cap M$ where M is a compact Lie group and G is a complexification of M . If E is a complex P -module then

$$H^\bullet(G/P, \mathcal{E}) = \sum K \otimes H^\bullet(\mathfrak{p}, \mathfrak{v}, \text{Hom}(K, E))$$

where K ranges over irreducible M -modules. This formula gives an expression of cohomology with coefficients in vector bundle in terms of relative Lie algebra cohomology, \mathfrak{p} stands for real

The relation between \mathcal{S} and \mathcal{Q} can be used to express cohomology of a graded L -module $N = \bigoplus_{m \geq m_0} N_m$ in terms of cohomology groups related to \mathcal{Q} . Recall that Corollary 5 permits us to reduce the calculation of the cohomology at hand to the calculation of the cohomology of the complex (of differential module)

$$N_c^\bullet{}_m = (N_m \otimes \mathcal{S}_0 \rightarrow N_{m+1} \otimes \mathcal{S}_1 \rightarrow \dots)$$

We can construct a differential vector bundle (a complex of holomorphic vector bundles \mathcal{N}^\bullet) over \mathcal{Q} in such a way that one obtains the above complex of modules considering holomorphic sections of vector bundles:

$$\begin{aligned} \mathcal{N}_c^\bullet{}_l &= (\dots \rightarrow N_{l-1} \otimes \mathcal{O}(-1) \rightarrow N_l \otimes \mathcal{O}(0) \rightarrow N_{l+1} \otimes \mathcal{O}(1) \rightarrow \dots) = \\ &= (\dots \rightarrow N_{l-1}(-1) \rightarrow N_l(0) \rightarrow N_{l+1}(1) \rightarrow \dots). \end{aligned} \quad (58)$$

We use here the notation $N(k) = N \otimes \mathcal{O}(k)$. Notice, that the construction of the complex of vector bundles depends on the choice of index l , but this dependence is very simple: $\mathcal{N}_c^\bullet{}_{l+1} = \mathcal{N}_c^\bullet{}_l(-1)$. The differential d_e is a multiplication by

$$e = \lambda^\alpha \theta_\alpha. \quad (59)$$

Let us assume that the modules N_i are also $\text{Spin}(10)$ -modules (more precisely, N is a module with respect of semidirect product of L and $\text{Spin}(10)$). Then vector bundles in the complex (58) are $\text{Spin}(10)$ -invariant; corresponding complex N_P of P -modules has the form

$$N_P = (\dots \rightarrow N_{l-1} \otimes \mu_{-1} \rightarrow N_l \otimes \mu_0 \rightarrow N_{l+1} \otimes \mu_1 \rightarrow \dots)$$

(Recall that $\mathcal{Q} = \text{SO}(10)/\text{U}(5)$ can be obtained also by means of taking quotient of complex spinor group $\text{Spin}(10, \mathbb{C})$ with respect to the subgroup P defined as a stabilizer of a point $\lambda_0 \in \mathcal{Q}$; see Section 2.3. The complex of P -modules comes from consideration of the complex of fibers over λ_0 .)

Let us consider hypercohomology of \mathcal{Q} with the coefficients in the complex $\mathcal{N}_l^\bullet = \mathcal{N}_c^\bullet{}_l$. These hypercohomology can be expressed in terms of the Dolbeault cohomology of \mathcal{N}_l^\bullet . Namely we should consider the bicomplex $\Omega^\bullet(\mathcal{N}_l^\bullet)$ of smooth Lie algebra of P and \mathfrak{v} stands for Lie algebra of $P \cap M$.

sections of the bundle of $(0, p)$ -forms with coefficients in \mathcal{N}_l^q . Two differentials are $\bar{\partial}$ and d_e . Hypercohomology $\mathbb{H}^i(\mathcal{Q}, \mathcal{N}_l)$ can be identified with cohomology of the total differential $\bar{\partial} + d_e$ in $\Omega^\bullet(\mathcal{N}_l^\bullet)$.

As usual we can analyze cohomology of the total differential by means of two spectral sequences whose E_2 terms are equal to $H^i(H^j(\Omega(\mathcal{N}_l), \bar{\partial}), d_e)$ and $H^i(H^j(\Omega(\mathcal{N}_l), d_e), \bar{\partial})$.

Proposition 15 *There is a long exact sequence of cohomology*

$$\begin{aligned} \dots \rightarrow H^i(N_c \mathbf{l}) \rightarrow \mathbb{H}^i(\mathcal{Q}, \mathcal{N}_l) \rightarrow H^{i-10}(N_h - \mathbf{8} - \mathbf{l}) \xrightarrow{\delta} \\ \xrightarrow{\delta} H^{i+1}(N_c \mathbf{l}) \rightarrow \dots \end{aligned} \quad (60)$$

Proof. It follows readily from equalities (56, 57) that nontrivial rows in E_2 of the first spectral sequence are $(H^0(\Omega(\mathcal{N}_l^\bullet), \bar{\partial}) = N_c \mathbf{l})$ and $(H^{10}(\Omega(\mathcal{N}_l^\bullet), \bar{\partial}) = N_h - \mathbf{8} - \mathbf{l})$. (We use the notations of Corollaries 5 and 6.) The operator δ is the differential in E_2 . To complete the proof we notice that this is the only non-vanishing differential in the spectral sequence. ■

We will be interested in graded L module $N = YM$; the corresponding graded differential vector bundle (complex of vector bundles) is denoted by \mathcal{YM} . Notice, that this bundle is $\text{Spin}(10)$ -invariant; it corresponds to the following representation of the group P :

$$L^2 + L^3 \otimes \mu_1 + L^4 \otimes \mu_2 + \dots \quad (61)$$

(As we have noticed there is a freedom in the construction of complex of vector bundle; the above formula corresponds to $l = 2$.)

Similarly starting with L module TYM one can define graded differential vector bundle \mathcal{TYM} ; it corresponds to the representation

$$L^3 \otimes \mu_1 + L^4 \otimes \mu_2 + \dots \quad (62)$$

of the group P .

More generally, we can consider the module

$$N = \bigoplus_{k \geq 0} N^k = \text{Sym}^j(YM) = \bigoplus_{k \geq 0} \text{Sym}^j(YM)^k$$

equipped with adjoint action of L . Corresponding complexes of vector bundles are denoted $\text{Sym}^j(\mathcal{YM})$. Symmetric algebra Sym is understood in the graded sense.

Similarly, we can define complexes of vector bundles $\text{Sym}^j(\mathcal{T}\mathcal{YM})$.

Let \mathcal{W}^* be the vector bundle on \mathcal{Q} induced from the representation W^* of P (see Section 2.3). It follows from (33) that there is an embedding $W^* \subset L^2 = YM^2 \subset YM$. From this we conclude that there is an embedding $\mathcal{W}^* \rightarrow \mathcal{YM}^\bullet$, where we consider \mathcal{W}^* as a graded vector bundle with one graded component W^* in grading 2 and zero differential (as one-term complex).

Proposition 16 *The embedding $\mathcal{W}^* \rightarrow \mathcal{YM}^\bullet$ is a quasi-isomorphism.*

We relegate the proof to the Appendix E.

Corollary 17 *The embedding of $\text{Sym}^i(\mathcal{W}^*)$ into $\text{Sym}^i(\mathcal{YM})$ is a quasi-isomorphism.*

Here $\text{Sym}^i(\mathcal{W}^*)$ is considered as graded vector bundle with grading $2i$. To deduce the corollary we use Künneth theorem.

We can reformulate Proposition 16 saying that the induced map of hypercohomology $\mathbb{H}^\bullet(\mathcal{W}^*) \rightarrow \mathbb{H}^\bullet(\mathcal{Q}, \mathcal{YM}_0^\bullet)$ is an isomorphism. Similarly, the map $\mathbb{H}^\bullet(\mathcal{Q}, \text{Sym}^i(\mathcal{W}^*)(l)) \rightarrow \mathbb{H}^\bullet(\text{Sym}^i(\mathcal{YM})_l^\bullet)$ is an isomorphism.

Using this statement and (60) we obtain

Corollary 18 *There is a long exact sequence of cohomology*

$$\begin{aligned} \dots \rightarrow H^i(\text{Sym}^j YM_c \iota) \rightarrow H^{i+l-2j}(\mathcal{Q}, \text{Sym}^j(\mathcal{W}^*)(2j-l)) \rightarrow H^{i-10}(\text{Sym}^j YM_h -\mathbf{8}-\iota) \xrightarrow{\delta} \\ \xrightarrow{\delta} H^{i+1}(\text{Sym}^j YM_c \iota) \rightarrow \dots \end{aligned} \quad (63)$$

Using Corollary 6 we can identify the cohomology $H^\bullet(\text{Sym}^j YM_h)$ with homology $H_\bullet(L, YM)$. Likewise $H^\bullet(\text{Sym}^j YM_c)$ is isomorphic to $H^\bullet(L, \text{Sym}(YM))$. This means that we can formulate (63) as a long exact sequence

$$\begin{aligned} \dots \rightarrow H^{i,l}(L, \text{Sym}^j YM) \rightarrow H^{i+l-2j}(\mathcal{Q}, \text{Sym}^j(\mathcal{W}^*)(2j-l)) \rightarrow \\ \rightarrow H_{2-i,l-\mathbf{8}}(L, \text{Sym}^j YM) \xrightarrow{\delta} H^{i+1,l}(L, \text{Sym}^j YM) \rightarrow \dots \end{aligned} \quad (64)$$

The hypercohomology $\mathbb{H}^\bullet(\mathcal{Q}, \text{Sym}^i(\mathcal{W}^*)(l))$ it is equal up to a shift in grading to the ordinary cohomology of the \mathcal{Q} vector bundle $\text{Sym}^i(\mathcal{W}^*)(l)$. Such cohomology can be computed via Borel-Weil-Bott theory.

Proposition 19

$$\begin{aligned} H^0(\mathcal{Q}, \text{Sym}^j(\mathcal{W}^*)(l)) &= [0, 0, 0, j, l-j], j, l-j \geq 0 \\ H^4(\mathcal{Q}, \text{Sym}^j(\mathcal{W}^*)(l)) &= [j-3, 0, 0, l+2, 0], l \geq -2, j \geq 3 \\ H^{10}(\mathcal{Q}, \text{Sym}^j(\mathcal{W}^*)(l)) &= [j, 0, 0, -8-l, 0], l \leq -8, j \geq 0 \end{aligned}$$

Straightforward inspection of the cohomology groups shows that the following groups are generated by Spin(10)-invariant elements : $\langle e \rangle = H^0(\mathcal{Q}, \text{Sym}^0(\mathcal{W}^)(0))$, $\langle c \rangle = H^4(\mathcal{Q}, \text{Sym}^3(\mathcal{W}^*)(-2))$, $\langle e' \rangle = H^{10}(\mathcal{Q}, \text{Sym}^0(\mathcal{W}^*)(-8))$.*

To analyze the super Poincaré invariant deformations we use Spin(10)-invariant part of exact sequence (64). It is easy to check that Spin(10)-invariant elements e, c are mapped into zero in this long exact sequence. This means that this long exact sequence splits into short exact sequences

$$\begin{aligned} &\text{if } i = 11, j = 0, l = 8 \text{ then} \\ &0 \rightarrow H^{i+l-2j-1}(\mathcal{Q}, \text{Sym}^j(\mathcal{W}^*)(2j-l))^{\text{Spin}(10)} \rightarrow \\ &\rightarrow H_{3-i, l-8}(L, \text{Sym}^j YM)^{\text{Spin}(10)} \xrightarrow{\delta} H^{i, l}(L, \text{Sym}^j YM)^{\text{Spin}(10)} \rightarrow 0 \\ &\text{otherwise} \\ &0 \rightarrow H_{3-i, l-8}(L, \text{Sym}^j YM)^{\text{Spin}(10)} \xrightarrow{\delta} H^{i, l}(L, \text{Sym}^j YM)^{\text{Spin}(10)} \rightarrow \\ &\rightarrow H^{i+l-2j}(\mathcal{Q}, \text{Sym}^j(\mathcal{W}^*)(2j-l))^{\text{Spin}(10)} \rightarrow 0 \end{aligned} \tag{65}$$

We see that Spin(10)-invariant elements of hypercohomology e, c contribute to cohomology $H^{0,0}(L, \mathbb{C})$, $H^{2,8}(L, \text{Sym}^3 YM)^{\text{Spin}(10)}$. The only non-trivial contribution corresponds to c and gives the infinitesimal deformation $\delta \mathcal{L}_{16}$ (42).

Proof of Theorem 11

We will give a proof of this theorem assuming that all infinitesimal supersymmetric deformations are given by Theorem 13. The key moment in the proof is the use of short exact sequence (65).

The operator δ in exact sequence (64) defines a map

$$\delta : H_1(L, \text{Sym}(YM)) \rightarrow H^2(L, \text{Sym}(YM))$$

whose kernel and cokernel are controlled by the exact sequence. We conclude that the space $\delta(H_1(L, \text{Sym}(YM))^{\text{Spin}(10)})$ has codimension one in $H^2(L, \text{Sym}(YM))^{\text{Spin}(10)}$. We will prove that the space of super Poincaré invariant deformations of equations of motion given by the formula (45) has the same codimension in $H^2(L, \text{Sym}(YM))^{\text{Spin}(10)}$ as $\delta(H_1(L, \text{Sym}(YM))^{\text{Spin}(10)})$; this gives a proof of the theorem 11. (The formula (45) specifies a supersymmetric deformation of Lagrangian. However, a deformation of Lagrangian function produces a deformation of equations of motions; this manifests in a map

$$var : H_0(YM, U(YM)) \rightarrow H^2(YM, U(YM)).$$

See Section 4 for more detail.)

Supersymmetry transformations θ_α act by derivations on Lie algebra YM .

¹⁵ From this we conclude that θ_α induce operators acting on objects constructed naturally (functorially) from YM . In particular they act on

$$H^i(YM, U(YM)) \xrightarrow{P} H_{3-i}(YM, U(YM)) \quad (66)$$

Here P denotes the Poincaré isomorphism (see Appendix A). The composition $\theta_1 \cdots \theta_{16}$ defines an operator in homology. We will use the notation A_k for this operator acting on k -dimensional homology:

$$A_k : H_k(YM, U(YM)) \xrightarrow{A} H_k(YM, U(YM))$$

In Section 3 we have interpreted the linear space $HH_0(U(YM), U(YM)) \cong H_0(YM, U(YM)) \cong H_0(YM, \text{Sym}(YM))$ as a linear space of infinitesimal deformations of action functionals in the reduced theory. Obviously the operator A_0 coincides with A defined in (45).

¹⁵This means that Lie algebra $\mathfrak{su}(16)$ acts on YM . Notice, however, that even part of $\mathfrak{su}(16)$ acts on YM trivially (supersymmetry transformations anticommute).

The maps A_k have an alternative description. Let N be an L -module. It is also a YM -module. Since homology is a covariant functor with respect to the Lie algebra argument there is a map $(i_*)_k : H_k(YM, N) \rightarrow H_k(L, N)$. Likewise there is a map in opposite direction on cohomology $(i^*)_k : H^k(L, N) \rightarrow H^k(YM, N)$. These observations enable us to define composition maps

$$\begin{aligned} T_k : H_k(YM, U(YM)) &\xrightarrow{(i_*)_k} H_k(L, U(YM)) \xrightarrow{\delta} H^{3-k}(L, U(YM)) \xrightarrow{(i^*)_{3-k}} \\ &\xrightarrow{(i^*)_{3-k}} H^{3-k}(YM, U(YM)) \xrightarrow{P} H_k(YM, U(YM)) \end{aligned}$$

Notice, that for the map i_2^* acts from $H^2(L, U(YM))$ into $H^2(YM, U(YM))$; we have shown in Section 4 that the elements in the image of this map correspond to supersymmetric deformations. The same arguments can be applied to the map i_k^* ; they lead to the conclusion that

$$T_k : H_k(YM, U(YM)) \rightarrow H_k(YM, U(YM))^{\mathfrak{su}\mathfrak{sh}}$$

(in other words, the image of T_k consists of supersymmetric elements). The map A_k obviously has the same feature, therefore it is natural to conjecture that the maps A_k and T_k coincide. To prove this conjecture we notice that the operators A_k and T_k can be defined for arbitrary L -module N as operators

$$H_k(YM, N) \rightarrow H_k(YM, N)^{\mathfrak{su}\mathfrak{sh}}$$

. Using free resolutions one can reduce the proof to the consideration of the module $N \cong U(L)$ where L acts on $U(L)$ by left multiplication (see [20] for details).

In general it is not easy to describe maps i_* and i^* . It is easier to analyze their restrictions to $\text{Spin}(10)$ -invariant elements. Let us consider maps $i_{*1} : H_1(YM, U(YM))^{\text{Spin}(10)} \rightarrow H_1(L, U(YM))^{\text{Spin}(10)}$ and $i_2^* : H^2(L, U(YM))^{\mathfrak{so}(10)} \rightarrow H^2(YM, U(YM))^{\mathfrak{so}(10) \times \mathfrak{su}\mathfrak{sh}} \subset H^2(YM, U(YM))^{\text{Spin}(10)}$. (Of course, $\text{Spin}(10)$ -invariance coincides with invariance with respect to corresponding Lie algebra $\mathfrak{so}(10)$; we use the language of Lie algebras to combine this invariance with $\mathfrak{su}\mathfrak{sh}$ -invariance.) One can prove the following

Lemma 20 *The maps i_{*1}, i_2^* are surjective.*

If we take this Lemma for granted we conclude that $A_1 : H_1(YM, U(YM))^{\mathfrak{so}(10)} \rightarrow H_1(YM, U(YM))^{\mathfrak{so}(10) \ltimes \mathfrak{su}(5)\eta}$ has one-dimensional co-kernel (of the same dimension as the co-kernel of the map δ).

The rather technical proof of the lemma (see [20]) is based on analysis of Serre-Hochschild spectral sequences associated with extension $YM \subset L$:

$$\begin{aligned} H_i(YM, \text{Sym}(YM)) \otimes \text{Sym}^j(S^*) &\Rightarrow H_{i+j}(L, \text{Sym}(YM)) \\ H^i(YM, \text{Sym}(YM)) \otimes \text{Sym}^j(S) &\Rightarrow H^{i+j}(L, \text{Sym}(YM)). \end{aligned}$$

Notice that the surjectivity of i_2^* has clear physical meaning: it can be interpreted as a statement that all super Poincaré invariant deformations in the sense of Section 4 are described by Theorem 12.

The above considerations gave us the information about the codimension of the image of the operator A_1 . To prove Theorem 11 we need information about the codimension of the image of A_0 . This information can be obtained from the results about operator A_1 by means of Connes differential

$$B : H_k(YM, U(YM)) \rightarrow H_{k+1}(YM, U(YM))$$

(see Appendix A). Using the fact that supersymmetries commute with the Connes differential we obtain that

$$A_{k+1}B = BA_k,$$

in particular, $A_1B = BA_0$.

We need the following

Lemma 21 *The map B defines a surjective map $H_0(YM, U(YM))^{\mathfrak{so}(10)} \rightarrow H_1(YM, U(YM))^{\mathfrak{so}(10)}$ with one-dimensional kernel generated by constants.*

Proof. The proof (see [16] and [20]) is based on a general theorem (see [13]) which asserts that the cohomology of B in $H_i(\mathfrak{g}, U(\mathfrak{g}))$ for positively graded \mathfrak{g} is trivial and generated by constants $\mathbb{C} \subset H_0(\mathfrak{g}, U(\mathfrak{g}))$. The rest follows from the information about homology of YM with coefficients in $U(YM)$ (see (refE:cohcomp)). ■

The proof of the statement that co-dimension of $\text{Im}(A_0)$ in the space of $\mathfrak{su}\mathfrak{sh}$ -invariant elements in $H_0(YM, U(YM))$ is equal to one easily follows from this lemma. We know that the image of map A_1 has co-dimension one in the space of $\mathfrak{su}\mathfrak{sh}$ -invariants. The operator B preserves $\mathfrak{so}(10) \ltimes \mathfrak{su}\mathfrak{sh}$ -invariant subspaces. If we write $H_0(YM, U(YM)) = \mathbb{C} + \underline{H}_0(YM, U(YM))$, the operator B admits the inverse: $B^{-1} : H_1(YM, U(YM))^{\text{Spin}(10)} \rightarrow \underline{H}_0(YM, U(YM))^{\text{Spin}(10)}$. The identity $A_1 B = B A_0$ implies that A_0 is equal to $B^{-1} A_1 B$, when restricted on $\underline{H}_0(YM, U(YM))^{\text{Spin}(10)}$. The claim follows from the corresponding statement for A_1 .

The reader should consult for missing details the references [16] and [17].

We have analyzed the case of reduced SYM theory. Similar considerations can be applied to the unreduced case.

First of all we should formulate the analog of Proposition 16. Let us notice that it follows from (35) that $W \otimes \mu_{-1} \subset L^3$, hence $W \subset L^3 \otimes \mu_1$. Using (62) we conclude that there is an embedding $\mathcal{W} \rightarrow \mathcal{T}\mathcal{YM}^\bullet$, where we consider \mathcal{W} as a graded vector bundle with one graded component in grading 3 that corresponds to P -module W and has zero differential.

Proposition 22 *The embedding of \mathcal{W} into $\mathcal{T}\mathcal{YM}^\bullet$ is a quasi-isomorphism.*

The proof will be given in Appendix E.

Using this proposition we can write down an exact sequence analogous to (64).

Corollary 23 *There is a long exact sequence connecting $H^k(L, U(TYM))$, $H_k(L, U(TYM))$ and hypercohomology:*

$$\begin{aligned} \cdots \rightarrow H_{3-i, a-8}(L, \text{Sym}^j(TYM)) &\xrightarrow{\delta} H^{i, a}(L, \text{Sym}^j(TYM)) \rightarrow \\ \rightarrow H^{i+a-3j}(\mathcal{Q}, \Lambda^j(\mathcal{W})(3j-a)) &\xrightarrow{\iota} H_{2-i, a-8}(L, \text{Sym}^j(TYM)) \rightarrow \cdots \end{aligned} \quad (67)$$

Again using Borel-Weil-Bott theorem we can calculate the cohomology of \mathcal{Q} with coefficients in vector bundles that enter this sequence.

Proposition 24

$$i \geq 0$$

$$\begin{aligned}
H^0(\mathcal{Q}, \mathcal{O}(i)) &= [0, 0, 0, 0, i], & H^{10}(\mathcal{Q}, \mathcal{O}(-8-i)) &= [0, 0, 0, i, 0], \\
H^0(\mathcal{Q}, \mathcal{W}(i+1)) &= [1, 0, 0, 0, i], & H^{10}(\mathcal{Q}, \mathcal{W}(-8-i)) &= [0, 0, 0, i, 1], \\
H^0(\mathcal{Q}, \Lambda^2(\mathcal{W})(2+i)) &= [0, 1, 0, 0, i], & H^{10}(\mathcal{Q}, \Lambda^2(\mathcal{W})(-8-i)) &= [0, 0, 1, i, 0], \\
H^9(\mathcal{Q}, \Lambda^2(\mathcal{W})(-6)) &= [0, 0, 0, 0, 0] \\
H^0(\mathcal{Q}, \Lambda^3(\mathcal{W})(3+i)) &= [0, 0, 1, 0, i], & H^{10}(\mathcal{Q}, \Lambda^3(\mathcal{W})(-7-i)) &= [0, 1, 0, i, 0], \\
H^1(\mathcal{Q}, \Lambda^3(\mathcal{W})(1)) &= [0, 0, 0, 0, 0] \\
H^0(\mathcal{Q}, \Lambda^4(\mathcal{W})(3+i)) &= [0, 0, 0, 1, i], & H^{10}(\mathcal{Q}, \Lambda^4(\mathcal{W})(-6-i)) &= [1, 0, 0, i, 0], \\
H^0(\mathcal{Q}, \Lambda^5(\mathcal{W})(3+i)) &= [0, 0, 0, 0, i], & H^{10}(\mathcal{Q}, \Lambda^5(\mathcal{W})(-5-i)) &= [0, 0, 0, i, 0],
\end{aligned} \tag{68}$$

To analyze super Poincaré invariant deformations of unreduced theory we should study $\text{Spin}(10)$ -invariant part of long exact sequence (67). As in reduced case $\text{Spin}(10)$ -part of the exact sequence splits into short exact sequences. More precisely if the indices (i, j, \mathbf{a}) belong to the set $\{(3, 0, 8), (4, 2, 12), (6, 5, 20)\}$ then we have the splitting

$$\begin{aligned}
0 \rightarrow H^{i+a-3j-1}(\mathcal{Q}, \Lambda^j(\mathcal{W})(3j-a))^{\text{Spin}(10)} &\rightarrow H_{3-i, a-8}(L, \text{Sym}^j(TYM))^{\text{Spin}(10)} \xrightarrow{\delta} \\
&\xrightarrow{\delta} H^{i, a}(L, \text{Sym}^j(TYM))^{\text{Spin}(10)} \rightarrow 0
\end{aligned}$$

$$\text{If } (i, j, \mathbf{a}) \in \{(0, 0, 0)(2, 3, 8)(3, 5, 12)\}$$

$$\begin{aligned}
0 \rightarrow H_{3-i, a-8}(L, \text{Sym}^j(TYM))^{\text{Spin}(10)} &\xrightarrow{\delta} H^{i, a}(L, \text{Sym}^j(TYM))^{\text{Spin}(10)} \rightarrow \\
&\rightarrow H^{i+a-3j}(\mathcal{Q}, \Lambda^j(\mathcal{W})(3j-a))^{\text{Spin}(10)} \rightarrow 0
\end{aligned}$$

and for all other (i, j, \mathbf{a})

$$H_{3-i, a-8}(L, \text{Sym}^j(TYM))^{\text{Spin}(10)} \xrightarrow{\delta} H^{i, a}(L, \text{Sym}^j(TYM))^{\text{Spin}(10)} \tag{69}$$

The $\text{Spin}(10)$ -invariant part of hypercohomology is six-dimensional, but only three-dimensional part of it, as the reader can see in (69), gives a contribution

to the cohomology H^0, H^2 and H^3 . The contribution to H^0 is not interesting; the contribution to H^2 gives the deformation $\delta\mathcal{L}_{16}$ and the contribution to H^3 is trivial at the level of infinitesimal deformations of equations of motion (but it gives a non-trivial deformation of L_∞ action of supersymmetry, hence the construction of Section 7 can give a non-trivial formal deformation).

The analogs of operators A_k and T_k can be defined in the situation at hand; again $A_k = T_k$.

The most technical part of the proof is hidden in the verification of the analog of Lemma 20.

Lemma 25 *The co-kernels of the maps*

$$i_{*1} : H_1(YM, U(TYM))^{\text{Spin}(10)} \rightarrow H_1(L, U(TYM))^{\text{Spin}(10)}$$

and

$$i_2^* : H^2(L, U(TYM))^{\mathfrak{so}(10)} \rightarrow H^2(YM, U(TYM))^{\mathfrak{so}(10) \ltimes \mathfrak{su}(2)}$$

have dimensions two and zero respectively.

The rest of the proof follows along the lines of the proof in reduced case. In particular one should use the analog of Lemma 21.

6 BV

Our considerations will be based on Batalin-Vilkovisky (BV) formalism. In this formalism a classical system is represented by an action functional S defined on an odd symplectic manifold M and obeying the classical Master equation

$$\{S, S\} = 0. \tag{70}$$

where $\{\cdot, \cdot\}$ stands for the odd Poisson bracket. Using an odd symplectic form $\omega = dz^A \omega_{AB} dz^B$ we assign to every even functional F an odd vector field ξ_F defined by the formula $\xi_F^A \omega_{AB} = \frac{\partial F}{\partial z^B}$. The form ω is invariant with respect to ξ_F . In particular we may consider an odd vector field $Q = \xi_S$; this field

obeys $[Q, Q] = 0$. Here $[\cdot, \cdot]$ stands for supercommutator. The solutions to the equations of motion (EM) are identified with zero locus of Q .

In an equivalent formulation of BV we start with an odd vector field Q obeying $[Q, Q] = 0$. We require the existence of Q -invariant odd symplectic form ω . Then we can restore the action functional from $Q^A \omega_{AB} = \frac{\partial S}{\partial z^B}$.

We say that a classical system is defined by means of an odd vector field Q obeying $[Q, Q] = 0$. In geometric language we are saying that a classical system is a Q -manifold. Fixing a vector field Q we specify equations of motion of our system, but we do not require that EM come from an action functional. If there exists a Q -invariant odd symplectic form we can say that our system comes from action functional S obeying classical Master equation $\{S, S\} = 0$. In this case we say that we are dealing with a Lagrangian system. In geometric language we can identify it with an odd symplectic Q -manifold.

Infinitesimal deformation of a classical system corresponds to a vector field ξ obeying $[Q, \xi] = 0$ (then $[Q + \xi, Q + \xi] = 0$ in the first order with respect to ξ). An infinitesimal deformation ξ is trivial if $\xi = [Q, \eta]$ because such a deformation corresponds to a change of variables (field redefinition) $z^A \rightarrow z^A + \eta^A$. Hence deformations of a classical system corresponding to vector field Q are labeled by cohomology of the space of vector fields $Vect(M)$. We assume that Q acts on vector fields by a commutator $\xi \rightarrow [Q, \xi]$ and denote the corresponding differential as \tilde{Q} .

The algebra of functions $C(M)$ on M can be considered as super commutative differential graded algebra with differential $\hat{Q} = Q^A \frac{\partial}{\partial z^A}$. The cohomology $\text{Ker} \hat{Q} / \text{Im} \hat{Q}$ can be identified with classical observables. In other words a classical observable is defined as a function O obeying $\hat{Q}O = 0$. Two classical observables O, O' are identified if the difference $O - O'$ is \hat{Q} of something. Similarly in the space Sol of solutions to EM (in the zero locus of Q) we should identify solutions x with $x + \delta x$ where $\delta x^A = Q^A(x + \delta) - Q^A(x)$, where δ is infinitesimally small. The space obtained by means of this identification is denoted by Sol / \sim .¹⁶

¹⁶More geometrically we can say that on the zero locus Sol of Q there exists a foliation \mathcal{F}_Q .

A classical system has many equivalent descriptions in BV-formalism. The simplest way to see this is to notice that a system with coordinates $(x^1, \dots, x^n, \xi_1, \dots, \xi_n)$, symplectic form $dx^i d\xi_i$ and action functional $a_{ij}x^i x^j$ is physically trivial. Here ξ_i and x^i have opposite parities and the matrix a_{ij} is nondegenerate.

Consider two Q -manifolds (M, Q) and (M', Q') . A map $f : M \rightarrow M'$ is called a Q -map if it agrees with action of Q 's (i.e. $\hat{Q}f^* = f^*\hat{Q}'$ where f^* is the homomorphism $C(M') \rightarrow C(M)$ induced by the map f). Such a map induces a map of observables (a homomorphism of cohomology groups $H(C(M'), Q') \rightarrow H(C(M), Q)$). If f defines an isomorphism between spaces of observables we say that f is a quasi-isomorphism. Under some additional requirements this isomorphism implies isomorphism of spaces of solutions Sol/\sim . Quasi-isomorphism should be considered as isomorphism of classical physical systems. However for Lagrangian systems one should modify the definition of physical equivalence, requiring that quasi-isomorphism is compatible with symplectic structure in some sense.

Let us consider the Taylor series decomposition

$$Q^a(x) = \sum_{b_1, \dots, b_n} Q_{b_1, \dots, b_n}^a x_1^{b_1} \dots x_n^{b_n}$$

of the coefficients of the vector field $Q = \sum Q^a \frac{\partial}{\partial x^a}$ in the neighborhood of the critical point. Here x^i are local coordinates in the patch, the critical point is located at $x = 0$. The coefficient Q_{b_1, \dots, b_n}^a of this expansion specifies an algebraic n -ary operation $\psi_n(s_1, \dots, s_n)$ on ΠT_0 (on the tangent space with reversed parity at $x = 0$). If Q is an odd vector field obeying $[Q, Q] = 0$, then the collection of operations satisfies some quadratic relations. If these relations are satisfied we say that $\{\psi_n\}_{n=1}^\infty$ specify a structure of L_∞ algebra on T_0 (see Appendix A for more detail). One can say that L_∞ algebra is a formal Q -manifold.¹⁷

The tangent vectors to Sol can be identified with the kernel $\text{Ker} \frac{\delta Q^A}{\delta z^B}$ of Q ; the leaves of \mathcal{F}_Q are tangent to the image $\text{Im} \frac{\delta Q^A}{\delta z^B}$. We identify two solutions belonging to the same leaf, hence Sol/\sim can be considered as the space of leaves of the foliation.

¹⁷Functions on a formal manifold are defined as series with respect to n commuting and m

In the case when the only nonzero coefficients are Q_b^a and Q_{b_1, b_2}^a the corresponding L_∞ algebra can be identified with a differential graded Lie algebra. The tensor Q_b^a corresponds to the differential and Q_{b_1, b_2}^a to the bracket.

An L_∞ homomorphism of L_∞ algebras is defined as a Q -map between two formal Q -manifolds. We can use this notion to define an L_∞ action of a Lie algebra on a Q -manifold M . Conventional action of a Lie algebra is a homomorphism of this Lie algebra into Lie algebra $Vect(M)$ of vector fields on M . If M is equipped with a vector field Q obeying $[Q, Q] = 0$ the commutator $[Q, \xi]$ defines the differential \tilde{Q} on the Lie algebra of vector fields $Vect(M)$. An L_∞ action of a Lie algebra \mathfrak{g} on (M, Q) as an L_∞ homomorphism of \mathfrak{g} to the differential graded algebra $(Vect(M), \tilde{Q})$.

An L_∞ action can be defined more explicitly. Notice that the action of a Lie algebra is specified by vector fields q_α , corresponding to generators e_α of \mathfrak{g} . The generators obey relations $[e_\alpha, e_\beta] = f_{\alpha\beta}^\gamma e_\gamma$, where $f_{\alpha\beta}^\gamma$ are the structure constants of \mathfrak{g} in the basis e_α . We define weak action of \mathfrak{g} requiring that this relation is valid up to Q -exact terms:

$$[q_\alpha, q_\beta] = f_{\alpha\beta}^\gamma q_\gamma + [Q, q_{\alpha\beta}] \quad (71)$$

It follows that we have a genuine Lie algebra action on observables and on Sol/\sim .

To define an L_∞ action of Lie algebra \mathfrak{g} we need not only $q_\alpha, q_{\alpha\beta}$, but also their higher analogs $q_{\alpha_1 \dots \alpha_i}$ obeying the relations similar to (71). This can be formalised as follows. One can consider $q_{\alpha_1 \dots \alpha_i}$ as components of linear maps

$$q^i : \text{Sym}^i(\Pi\mathfrak{g}) \rightarrow \Pi Vect(M) \quad (72)$$

They can be assembled into a vector field q on $\Pi\mathfrak{g} \times M$. A choice of a basis in \mathfrak{g} defines coordinates on $\Pi\mathfrak{g}$. In such coordinates the i -th Taylor coefficient coincides with the map q^i . The coordinates on $\Pi\mathfrak{g}$ will be denoted by c^α ; they can be identified with ghost variables for the Lie algebra \mathfrak{g} . One can consider q as a vector field on M depending on ghost variables.

anticommuting variables.

Let us introduce a super-commutative differential algebra $C^\bullet(\mathfrak{g})$ as the algebra of polynomial functions of ghost variables c^α with the differential

$$d_{\mathfrak{g}} = \frac{1}{2} f_{\alpha\beta}^\gamma c^\alpha c^\beta \frac{\partial}{\partial c^\gamma}. \quad (73)$$

Odd ghosts correspond to even generators, even ghosts correspond to odd generators. The Lie group cohomology is defined as the cohomology of $d_{\mathfrak{g}}$.

The collection (72) defines a L_∞ action if the ghost dependent vector field q satisfies

$$d_{\mathfrak{g}}q + [Q, q] + \frac{1}{2}[q, q] = 0. \quad (74)$$

Notice, that instead of q we can consider ghost dependent vector field $\tilde{q} = Q + q$; in terms of this field (74) takes the form

$$d_{\mathfrak{g}}\tilde{q} + \frac{1}{2}[\tilde{q}, \tilde{q}] = 0 \quad (75)$$

The notion of L_∞ action is a particular case of the notion of L_∞ module. Recall that a \mathfrak{g} -module where \mathfrak{g} is a Lie algebra can be defined as a homomorphism of \mathfrak{g} in the Lie algebra of linear operators acting on vector space N . (In other words, \mathfrak{g} -module is the same as linear representation of \mathfrak{g} .) If N is a differential module (i.e. N is a \mathbb{Z}_2 graded space equipped with an odd linear operator d obeying $d^2 = 0$) the space of linear operators on N is a differential Lie algebra. A structure of $L_\infty \mathfrak{g}$ module on N is an L_∞ homomorphism of \mathfrak{g} into this differential Lie algebra. This structure can be described as a polynomial function q of ghosts c^α taking values in the space of linear operators on N and obeying relation:

$$d_{\mathfrak{g}}q + [d, q] + \frac{1}{2}[q, q] = 0. \quad (76)$$

(This is the relation (74) where Q is replaced by the differential d .)

We can define cohomology $H_\bullet^\bullet(N) = H^\bullet(\mathfrak{g}, N)$ of the Lie algebra \mathfrak{g} with coefficients in $L_\infty \mathfrak{g}$ -module N as cohomology of the differential

$$d_c = d_{\mathfrak{g}} + q + d = \frac{1}{2} f_{\alpha\beta}^\gamma c^\alpha c^\beta \frac{\partial}{\partial c^\gamma} + \sum_k \frac{1}{k!} q_{\alpha_1, \dots, \alpha_k} c^{\alpha_1} \dots c^{\alpha_k} + d. \quad (77)$$

acting on the space of N -valued functions of ghosts (on the tensor product $C^\bullet(\mathfrak{g}) \otimes N$). It follows immediately from (76) that d_c is a differential. Conversely, if the expression (77) is a differential q specifies an L_∞ action.

To define homology of the Lie algebra \mathfrak{g} with coefficients in L_∞ module N we use the differential d_h acting on N -valued polynomial functions of ghost variables c_α (on the tensor product $\text{Sym} \Pi \mathfrak{g} \otimes N$). This differential can be obtained from d_c by means of substitution of the derivation with respect to c_α instead of multiplication by c^α and of the multiplication by c_α instead of derivation with respect to c^α :

$$d_h = \frac{1}{2} f_{\alpha\beta}^\gamma c_\gamma \frac{\partial}{\partial c_\alpha} \frac{\partial}{\partial c_\beta} + \sum_k \frac{1}{k!} q_{\alpha_1, \dots, \alpha_k} \frac{\partial}{\partial c_{\alpha_1}} \cdots \frac{\partial}{\partial c_{\alpha_k}} + d \quad (78)$$

If M is an odd symplectic manifold, then the definition of a Hamiltonian L_∞ action is obvious. In the formula (74) we just replace the vector field q by a function and the commutator by the Poisson bracket. A Hamiltonian L_∞ symmetry of classical BV action functional S can be specified by a function of ghosts and fields (by an element $\sigma \in C^\bullet(\mathfrak{g}) \otimes C^\infty(M)$). This element should obey the equation

$$d_{\mathfrak{g}} \sigma + \{S, \sigma\} + \frac{1}{2} \{\sigma, \sigma\} = 0. \quad (79)$$

Introducing a function $\hat{S} = \sigma + S$ we can rewrite (79) in the form

$$d_{\mathfrak{g}} \hat{S} + \frac{1}{2} \{\hat{S}, \hat{S}\} = 0. \quad (80)$$

In many interesting situations an action of a Lie algebra on shell (on the space Sol/\sim) can be lifted to an L_∞ action off shell. Conversely any L_∞ action of Lie algebra (or, more generally, any weak action) on a Q-manifold (off-shell action) generates ordinary Lie algebra action on shell.

More generally, we can consider L_∞ \mathfrak{g} -module N . We say that the structure of L_∞ module on N is Hamiltonian if N can be equipped with \mathfrak{g} -invariant inner product. (We say that the inner product is \mathfrak{g} -invariant if the function q specifying L_∞ structure takes values in the Lie algebra of linear operators

on N that preserve the inner product.)¹⁸ We can generalize the notion of Hamiltonian L_∞ action allowing ghost-dependent inner products. (In other words, we can assume that the inner product takes values in $\text{Sym}(\Pi\mathfrak{g})$.)

Let us come back to the general theory of deformations in BV-formalism. Recall that infinitesimal deformations of solution to the classical Master equation (70) are labeled by observables (by cohomology $H(C(M), Q)$ of Q on the space $C(M)$). Of course, every deformation of S induces a deformation of Q and we have a homomorphism of corresponding cohomology groups $H(C(M), Q) \rightarrow H(\text{Vect}(M), \tilde{Q})$.

Let us analyze the classification of deformations of a classical system preserving a symmetry of the system. Let us assume that the system is described by an odd vector field Q obeying $\{Q, Q\} = 0$ on supermanifold M and that the symmetry is specified by L_∞ action of the Lie algebra of \mathfrak{g} . This means that we can consider the differential module $(\text{Vect}(M), \tilde{Q})$ as L_∞ \mathfrak{g} -module. We would like to deform simultaneously the vector field Q and the L_∞ action specified by the ghost dependent vector field q . *We will show that the infinitesimal deformations are classified by elements of cohomology $H^\bullet(\mathfrak{g}, (\text{Vect}(M), \tilde{Q}))$ of Lie algebra \mathfrak{g} with coefficients in differential L_∞ \mathfrak{g} -module $(\text{Vect}(M), \tilde{Q})$.* To prove this statement we notice that Q and q are combined in the ghost dependent vector field \tilde{q} , hence the deformation we are interested in can be considered as the deformation of \tilde{q} that preserves the relation (75). In other words this deformation should obey

$$d_{\mathfrak{g}}\delta\tilde{q} + [\tilde{q}, \delta\tilde{q}] = 0. \quad (81)$$

This condition means that $\delta\tilde{q}$ is a cocycle specifying an element of cohomology group at hand (see (77)). It is easy to see that cohomologous cocycles specify equivalent deformations.

¹⁸If we have an odd symplectic Q -manifold M we can take as N the L_∞ algebra constructed as the Taylor decomposition of Q in Darboux coordinates in the neighborhood of a point belonging to the zero locus of Q . This algebra is equipped with odd inner product coming from the odd symplectic form. A Hamiltonian L_∞ action on M generates a Hamiltonian L_∞ action on N .

It is important to emphasize that commutation relations of the new symmetry generators are deformed (even if we have started with genuine action of \mathfrak{g} we can obtain a weak action after the deformation). Nevertheless on shell, i.e. after restriction to Sol/\sim , commutation relations do not change.

As we have said the cohomology $H^\bullet(\mathfrak{g}, (Vect(M), \tilde{Q}))$ describes L_∞ deformations of Q and L_∞ action. Notice, that two different L_∞ actions can induce the same Lie algebra action on shell. It is easy to see that only the ghost number one components enter in the expressions for generators of Lie algebra symmetries on shell.

It is important to emphasize that analyzing deformations in BV formulation we can chose any of physically equivalent classical systems (the cohomology we should calculate is invariant with respect to quasi-isomorphism).

We have analyzed the deformations of Q (of equations of motion) preserving the symmetry. Very similar consideration can be applied in Lagrangian formalism. In this case we start with the functional $\hat{S} = \sigma + S$ combining the classical BV functional S and Hamiltonian L_∞ symmetry. We should deform this functional preserving the relation (80). We see that that the infinitesimal deformation obeys

$$d_{\mathfrak{g}}\delta\hat{S} + \{\hat{S}, \delta\hat{S}\} = 0.$$

Interpreting this equation as a cocycle condition we obtain the following statement.

Proposition 26 *Let us consider a BV action functional S on an odd symplectic manifold M together with Hamiltonian L_∞ action of Lie algebra \mathfrak{g} (i.e. with a functional σ of fields and ghosts such that $\hat{S} = \sigma + S$ obeys (80)). Then the algebra $C^\infty(M)$ can be considered as a differential L_∞ \mathfrak{g} -module (the differential is defined as a Poisson bracket with S). Infinitesimal deformations of BV-action functional and Hamiltonian L_∞ action are governed by the Lie algebra cohomology of \mathfrak{g} with coefficients in this module.*

Let us describe some formulations of 10D SUSY YM in BV formalism. For simplicity we will restrict ourself to the theory reduced to a point.

In component formalism besides fields A_i, χ^α , antifields A_i^*, χ_α^* we have ghosts c and anti-fields for ghosts c^* (all of them are $n \times n$ matrices). The BV action functional has the form

$$\mathcal{L}_{BVSYM} = \text{tr} \left(\frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} \Gamma_{\alpha\beta}^i \chi^\alpha \nabla_i \chi^\beta + \nabla_i c A_i^* + \chi^\alpha [c, \chi_\alpha^*] + \frac{1}{2} [c, c] c^* \right) \quad (82)$$

The corresponding vector field Q is given

$$\begin{aligned} Q(A_i) &= -\nabla_i c \\ Q(\psi^\alpha) &= [c, \psi^\alpha] \\ Q(c) &= \frac{1}{2} [c, c] \\ Q(c^*) &= \sum_{i=1}^{10} \nabla_i A^{*i} + \sum_{\alpha} [\psi^\alpha, \psi_\alpha^*] + [c, c^*] \\ Q(A^{*m}) &= -\sum_{i=1}^{10} \nabla_i F_{im} + \frac{1}{2} \sum_{\alpha\beta} \Gamma_{\alpha\beta}^m [\psi^\alpha, \psi^\beta] - [c, A^{*m}] \\ Q(\psi_\alpha^*) &= -\sum_{i=1}^{10} \sum_{\beta} \Gamma_{\alpha\beta}^i \nabla_i \psi^\beta - [c, \psi_\alpha^*] \end{aligned} \quad (83)$$

Another possibility is to work in the formalism of pure spinors.

Let $S = \mathbb{C}^{16}$ be a 16-dimensional complex vector space with coordinates $\lambda^1, \dots, \lambda^{16}$. Denote by $C\mathcal{Q}$ a cone of pure spinors in S defined by equation

$$\Gamma_{\alpha\beta}^i \lambda^\alpha \lambda^\beta = 0 \quad (84)$$

and by $\mathcal{S} = \mathbb{C}[\lambda^1, \dots, \lambda^{16}] / \Gamma_{\alpha\beta}^i \lambda^\alpha \lambda^\beta$ the space of polynomial functions on $C\mathcal{Q}$.

The fields in this formulation are elements $A(\lambda, \theta) \in \mathcal{S} \otimes \Lambda[\theta^1, \dots, \theta^{16}] \otimes \text{Mat}_N$; they can be considered as Mat_N -valued polynomial super-functions on $C\mathcal{Q} \times \Pi S$. We define differential d acting on these fields by the formula $d = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha}$. Using the terminology of Section 2.1 we can identify the space of fields with tensor product of reduced Berkovits algebra B_0 and Mat_N .

The vector field Q on the space of fields is given by the formula

$$\delta_Q A = dA + \frac{1}{2} \{A, A\}. \quad (85)$$

This vector field specifies a classical system quasi-isomorphic to the classical system corresponding to the action functional (82) (see [18]).

The odd symplectic form on the space of fields is given by the formula:

$$\omega(\delta A_1, \delta A_2) = \text{tr}(\delta A_1 \delta A_2) \quad (86)$$

The trace tr is nontrivial only on $\mathcal{S}_3 \otimes \Lambda^5[\theta^1, \dots, \theta^{16}]$. Denote Γ be the only $\text{Spin}(10)$ invariant element in $\mathcal{S}_3 \otimes \Lambda^5[\theta^1, \dots, \theta^{16}]$. Let p be the only $\text{Spin}(10)$ -invariant projection on the span $\langle \Gamma \rangle$. Then $p(a) = \text{tr}(a)\Gamma$. This definition fixes tr up to a constant. (The trace at hand was introduced in [2], where more explicit formula was given.) Notice that ω is a degenerate closed two-form. We factorize the space of fields with respect to the kernel of ω and consider ω as a symplectic form on the quotient.

In the BV-formalism equations of motion can be obtained from the action functional

$$S(A) = \text{tr}(AdA + \frac{2}{3}A^3)$$

, obeying the classical Master equation $\{S, S\} = 0$ (recall that we factorize the space of fields with respect to $\text{Ker}\omega$ and S descends to the quotient). The vector field Q specified by the formula (85) corresponds to this action functional.

The BV formulation of unreduced SYM theory in terms of pure spinors is similar. The basic field $A(x, \lambda, \theta)$ where x is a ten-dimensional vector is matrix-valued. The differential d is defined by the formula $d = \lambda^\alpha (\frac{\partial}{\partial \theta^\alpha} + \Gamma_{\alpha\beta}^i \theta^\beta \frac{\partial}{\partial x^i})$. In the terminology of Section 2.1 the space of fields is a tensor product of Berkovits algebra B and Mat_N . The expressions for action functional and odd symplectic form remain the same, but tr includes integration over ten-dimensional space.

19

¹⁹To establish the relation to the superspace formalism we recall that in $(10|16)$ dimensional superspace (x^n, θ^α) SYM equations together with constraints can be represented in the form

$$F_{\alpha\beta} = 0 \quad (87)$$

where $F_{\alpha\beta} = \{\nabla_\alpha, \nabla_\beta\} - \Gamma_{\alpha\beta}^i \nabla_i$, $\nabla_\alpha = D_\alpha + A_\alpha$, $D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \Gamma_{\alpha\beta}^i \theta^\beta \frac{\partial}{\partial x^i}$. It follows from these equations that the covariant derivatives $\nabla(\lambda) = \lambda^\alpha \nabla_\alpha$ obey $[\nabla(\lambda), \nabla(\lambda)] = 0$ if λ is a pure spinor. This allows us interpret Yang-Mills fields as degree one components of $A(x, \theta, \lambda)$. Degree zero components of $A(x, \theta, \lambda)$ correspond to ghosts. Degree two components correspond to antifields, degree three to antifields for ghosts. Components of higher degree belong to the kernel of ω and can be disregarded (see [2] and [18] for detail).

Notice that in component version of BV formalism the standard supersymmetry algebra acts on shell, but off shell we have weak action of this algebra (commutation relations are satisfied up to Q -trivial terms). In pure spinor formalism we have genuine action of supersymmetry algebra, but the form ω is not invariant with respect to supersymmetry transformations. (However, the corrections to this form are Q -trivial.)

We will show that the weak action of supersymmetry algebra can be extended to L_∞ action (Appendix C). Moreover, in Appendix D we will prove that for appropriate choice of this action it will be compatible with odd symplectic structure.

Let us apply our general considerations to 10D SUSY YM reduced to a point. In Section 3 we described SUSY deformations of this action functional in component formalism. Now we will rewrite these deformations in BV formalism. Moreover, we will be able to write down also the deformed supersymmetry.

Let us start with BV description of the theory based on the Lagrangian (82). A vector field ξ on the underlying space is completely characterised by the values of the on the generators of the algebra of functions. We will refer to these values as to components. As the generators can be naturally combined in matrices, the components of the vector fields are also matrix-valued. The vector fields of supersymmetries θ_α in the matrix space description have the following components (we omit matrix indices):

$$\begin{aligned}\theta_\alpha A^i &= \Gamma_{\alpha\beta}^i \chi^\beta \\ \theta_\alpha \chi^\beta &= \Gamma_{\alpha}^{\beta ij} [A_i, A_j].\end{aligned}$$

The components description of the vector fields D_i and G_α is

$$D_i A_j = [A_i, A_j], \quad D_i \chi^\alpha = [A_i, \chi^\alpha]$$

and

$$G_\alpha A_j = [\chi_\alpha, A_j], \quad G^\alpha \chi^\beta = [\chi^\alpha, \chi^\beta].$$

In this setup we have the identities

$$\begin{aligned} [\theta_\alpha, \theta_\beta] - \Gamma_{\alpha\beta} D_i &= [Q, \eta_{\alpha\beta}], \\ [\theta_\alpha, D_i] - \Gamma_{\alpha\beta i} G_{\chi^\beta} &= [Q, \eta_{\alpha i}] \end{aligned} \tag{88}$$

Here

$$\begin{aligned} \eta_{\alpha\beta} \chi^\gamma &= 2P_{\alpha\beta}^{\gamma\delta} \chi_\delta^*, \\ \eta_{\alpha i} A_j &= C_{\alpha i}^{\beta j} \chi_\beta^*, \quad \eta_{\alpha i} \chi^\beta = -C_{\alpha i}^{\beta j} A_j^* \end{aligned}$$

The tensors $P_{\alpha\beta}^{\gamma\delta}$ and $C_{\alpha i}^{\beta j}$ are proportional to $\Gamma_{\alpha\beta}^{i_1, \dots, i_5} \Gamma_{i_1, \dots, i_5}^{\gamma\delta}$ and to $\Gamma_{\alpha i}^{\beta j}$ respectively. We have described in Section 3 an infinite family of SUSY deformations (45). It is easy to write down the terms q and q_α in the corresponding cocycle. It is obvious that q is a Hamiltonian vector field corresponding to the functional $\delta\mathcal{L}$ given by the formula (45). To find the functional σ_α generating the Hamiltonian vector field $q_\alpha c^\alpha$ we should calculate $\theta_\alpha \delta\mathcal{L}$ and use (23). The calculation of $\theta_\alpha \delta\mathcal{L}$ repeats the proof of the supersymmetry of the deformation

(9) and leads to the following result:

$$\begin{aligned}
\theta_\alpha \delta \mathcal{L} &= \text{tr}(\theta_\alpha \theta_1 \dots \theta_{16} G) = \\
&\text{tr}\left(\sum_{\gamma=1}^{\alpha-1} (-1)^{\gamma-1} \Gamma_{\alpha\gamma}^k \theta_1 \dots \theta_{\gamma-1} \tilde{Q}(\eta_{\alpha\gamma}) \theta_{\gamma+1} \dots \theta_{16} G\right) + \\
&+ \text{tr}\left(\sum_{\gamma=1}^{\alpha-1} (-1)^{\gamma-1} \Gamma_{\alpha\gamma}^k \theta_1 \dots \theta_{\gamma-1} D_k \theta_{\gamma+1} \dots \theta_{16} G\right) + \\
&+ \frac{1}{2} \text{tr}\left((-1)^{\alpha-1} \Gamma_{\alpha\alpha}^k \theta_1 \dots \theta_{\alpha-1} \tilde{Q}(\eta_{\alpha\alpha}) \theta_{\alpha+1} \dots \theta_{16} G\right) + \\
&+ \frac{1}{2} \text{tr}\left((-1)^{\alpha-1} \Gamma_{\alpha\alpha}^k \theta_1 \dots \theta_{\alpha-1} D_k \theta_{\alpha+1} \dots \theta_{16} G\right) = \\
&= Q \sum_{\gamma=1}^{\alpha-1} \Gamma_{\alpha\gamma}^k \text{tr}(\theta_1 \dots \theta_{\gamma-1} \eta_{\alpha\gamma} \theta_{\gamma+1} \dots \theta_{16} G) + \\
&+ Q \frac{1}{2} \Gamma_{\alpha\alpha}^k \text{tr}(\theta_1 \dots \theta_{\alpha-1} \eta_{\alpha\alpha} \theta_{\alpha+1} \dots \theta_{16} G) + \\
&+ Q \sum_{\gamma=1}^{\alpha-1} \sum_{\gamma'=1}^{\gamma-1} (-1)^{\gamma+\gamma'} \Gamma_{\alpha\gamma}^k \text{tr}(\theta_1 \dots \theta_{\gamma'-1} \eta_{\alpha k} \theta_{\gamma'+1} \dots \hat{\theta}_\gamma \dots \theta_{16} G) + \\
&+ Q \frac{1}{2} \sum_{\gamma=1}^{\alpha-1} (-1)^{\alpha+\gamma} \Gamma_{\alpha\alpha}^k \text{tr}(\theta_1 \dots \theta_{\gamma-1} \eta_{\alpha k} \theta_{\gamma+1} \dots \hat{\theta}_\alpha \dots \theta_{16} G) + \\
&+ \sum_{\gamma=1}^{\alpha-1} \frac{\partial}{\partial x^k} \text{tr}\left((-1)^{\gamma-1} \Gamma_{\alpha\gamma}^k \theta_1 \dots \hat{\theta}_\gamma \dots \theta_{16} G\right) + \\
&+ \frac{1}{2} \frac{\partial}{\partial x^k} \text{tr}\left((-1)^{\alpha-1} \Gamma_{\alpha\alpha}^k \theta_1 \dots \hat{\theta}_\alpha \dots \theta_{16} G\right).
\end{aligned} \tag{89}$$

The roof “^” marks the symbol that ought to be omitted in the formula. If subscript in θ_γ is out of range $[1, 16]$ then θ_γ must be omitted. From this computation we conclude that for Hamiltonian of the vector field q_α as a function of G is

$$\begin{aligned}
&\sum_{\gamma=1}^{\alpha-1} \Gamma_{\alpha\gamma}^k \text{tr}(\theta_1 \dots \theta_{\gamma-1} \eta_{\alpha\gamma} \theta_{\gamma+1} \dots \theta_{16} G) + \\
&+ \frac{1}{2} \Gamma_{\alpha\alpha}^k \text{tr}(\theta_1 \dots \theta_{\alpha-1} \eta_{\alpha\alpha} \theta_{\alpha+1} \dots \theta_{16} G) + \\
&+ \sum_{\gamma=1}^{\alpha-1} \sum_{\gamma'=1}^{\gamma-1} (-1)^{\gamma+\gamma'} \Gamma_{\alpha\gamma}^k \text{tr}(\theta_1 \dots \theta_{\gamma'-1} \eta_{\alpha k} \theta_{\gamma'+1} \dots \hat{\theta}_\gamma \dots \theta_{16} G) + \\
&+ \frac{1}{2} \sum_{\gamma=1}^{\alpha-1} (-1)^{\alpha+\gamma} \Gamma_{\alpha\alpha}^k \text{tr}(\theta_1 \dots \theta_{\gamma-1} \eta_{\alpha k} \theta_{\gamma+1} \dots \hat{\theta}_\alpha \dots \theta_{16} G)
\end{aligned} \tag{90}$$

7 Formal SUSY deformations

We have analyzed infinitesimal SUSY deformations of reduced and unreduced SUSY YM theory. One can prove that all of these deformations can be extended to formal deformations (i.e. there exist SUSY deformations represented as formal series with respect to parameter ϵ and giving an arbitrary infinitesimal deformation in the first order with respect to ϵ). We will sketch the proof of this fact in present section.

We have seen in Section 6 that there is a large class of infinitesimal supersymmetric deformations that have a form $\theta_1 \dots \theta_{16} G$. We will start with the proof that all these infinitesimal deformations can be extended to formal deformations.

We will consider more general situation when we have any action functional in BV formalism that is invariant with respect to L_∞ action of SUSY. As follows from Appendix D our considerations can be applied to ten- dimensional SYM theory.

The SUSY Lie algebra has m even commuting generators X_1, \dots, X_m and n odd generators τ_1, \dots, τ_n obeying relations

$$[\tau_\alpha, \tau_\beta] = \Gamma_{\alpha\beta}^i X_i.$$

In the definition of L_∞ -action of \mathfrak{g} we use the algebra $C^\bullet(\mathfrak{g})$ of functions of corresponding ghosts. In our case this algebra is the algebra

$$K = \mathbb{C}[[t^1, \dots, t^n]] \otimes \Lambda[\xi^1, \dots, \xi^m]. \quad (91)$$

The odd variables ξ^1, \dots, ξ^m are the ghosts for even generators (space-time translations), the even variables are the ghosts for odd generators. The algebra K is equipped with the differential

$$d = \Gamma_{\alpha\beta}^i t^\alpha t^\beta \frac{\partial}{\partial \xi_i},$$

where $\Gamma_{\alpha\beta}^i$ are the structure constants of the supersymmetry algebra.

The L_∞ action can be described by an element $\hat{S} \in A = K \otimes C^\infty(M)$, where M is the space of fields (in other words \hat{S} is a function of ghost variables t^i, ξ^α and fields).

The equation (80) for \hat{S} takes the form

$$d\hat{S} + \frac{1}{2}\{\hat{S}, \hat{S}\} = 0. \quad (92)$$

A solution to this equation gives us a solution S to the BV Master equation (obtained if we assume that ghost variables are equal to zero) and L_∞ action of supersymmetries preserving S . We would like to construct a formal deformation of such a solution, i.e. we would like to construct a formal power series $\hat{S}(\epsilon)$ with respect to ϵ obeying the equation (92) and giving the original solution for $\epsilon = 0$. We will start with a construction of the solution of the equation for infinitesimal deformation

$$dH + \{\hat{S}, H\} = 0. \quad (93)$$

If we know the solution of the equation (93) for every \hat{S} we can find the deformation solving the equation

$$\frac{d\hat{S}(\epsilon)}{d\epsilon} = H(\hat{S}(\epsilon)). \quad (94)$$

To solve the equation (93) we construct a family of functions F^k defined by inductive formula

$$F^{k+1} = \frac{1}{t_{k+1}} \left(d_{k+1} F^k + \{\hat{S}^k, F^k\} \right). \quad (95)$$

where d_k is defined as $\sum_{\alpha\beta \leq k} \Gamma_{\alpha\beta}^i t^\alpha t^\beta \frac{\partial}{\partial \xi_i}$. We assume that F^k and \hat{S}^k do not depend on t^{k+1}, \dots, t^n and \hat{S}^k coincides with \hat{S} if $t^{k+1} = \dots = t^n = 0$. We impose also an initial condition F^0 obeying $\{\hat{S}_i, F^0\} = 0$ where $\hat{S}_i = \frac{\partial \hat{S}}{\partial \xi_i}|_0$. We will see that F^n is a solution of the equation (93); this allows us to take $H = F^n$. To prove this fact we should give geometric interpretation of (95). First of all we notice that the solutions of (93) are cocycles of the differential $d_S = d + \{\hat{S}, \cdot\}$ acting on the algebra A of functions of ghosts $t^1, \dots, t^n, \xi^1, \dots, \xi^m$ and fields. We consider differential ideal I_k of this algebra defined as set of functions that

vanish if $t^1 = \dots = t^k = 0$ (in other words, I_k is generated by t^1, \dots, t^k) and the quotient A_k of the algebra A with respect to this ideal. The differential algebra A_k can be interpreted as the algebra of functions depending on ghosts $t^1, \dots, t^k, \xi^1, \dots, \xi^m$ and fields. The inductive formula (95) gives a map of A_k into A_{k+1} that descends to cohomology. To construct this map we notice that the embedding $I_{k+1} \subset I_k$ generates a short exact sequence

$$0 \rightarrow I_k/I_{k+1} \rightarrow A_{k+1} \rightarrow A_k \rightarrow 0.$$

The ideal I_k/I_{k+1} of the algebra A_{k+1} is generated by one element t^{k+1} . This means we can rewrite the exact sequence in the form

$$0 \rightarrow A_{k+1} \xrightarrow{\cdot t^{k+1}} A_{k+1} \rightarrow A_k \rightarrow 0,$$

where the map $A_{k+1} \rightarrow A_{k+1}$ is a multiplication by t^{k+1} . The boundary map in the corresponding exact cohomology sequence gives (95). The condition imposed on F^0 means that F^0 is a cocycle in A_0 .

For every admissible F^0 we have constructed $H(S)$ as a solution of (93); we have used this solution to construct formal deformation by means of (94).

This fairly simple description of supersymmetric deformations has one obvious shortcoming. The Poincaré invariance is hopelessly lost in the formula (95) even if we start with Poincaré invariant F^0 . This can be fixed if we work in the euclidean signature. The algebra A contains a subalgebra $A_{\text{SO}(m)}$ of $\text{SO}(m)$ -invariant elements. The vector field $H(F^0)$, restricted on $A_{\text{SO}(m)}$ can be replaced by

$$H^{\text{SO}(m)} = \frac{1}{\text{vol}(\text{SO}(m))} \int_{\text{SO}(m)} H^g dg$$

- the average of the g -rotated element H over $\text{SO}(m)$. It can be proved by other means that $H_{F^0}^{\text{SO}(m)}$ is nonzero if F^0 is Poincaré-invariant. The above prescription can be formulated also in more algebraic form where Euclidean signature is unnecessary. We decompose A into direct sum of irreducible representations of $\text{SO}(m)$ and leave only $\text{SO}(m)$ invariant part of H .

Let us make a connection with Section 3.

We start with identifications. The odd symplectic manifold M coincides with the space of fields in the maximally super-symmetric Yang-Mills theory in Batalin-Vilkovisky formalism (we can consider both reduced case when $n = 16, m = 0$ and unreduced case when $n = 16, m = 10$). It can be shown that the supersymmetry action can be extended to an L_∞ action, whose generating function satisfies equation (92); see Appendix D.

Let us start with a Poincaré invariant $F^0 = G$ as described in Section 3. The l'Hôpital's rule applied to $H = F^n$ shows that its leading term coincides with (45). This means that infinitesimal SUSY deformations of the form $\text{tr}\theta_1 \cdots \theta_{16}G$ can be extended to formal deformations. In reduced case this logic can be applied to arbitrary Poincaré invariant G , in unreduced case we should consider local gauge covariant G to obtain SUSY deformation.

There exists only one infinitesimal deformation $\delta\mathcal{L}_{16}$ that does not have the form $\text{tr}\theta_1 \cdots \theta_{16}G$ (Theorem 11 and Theorem 12).²⁰ One can prove that this deformation also can be extended to formal deformation together with L_∞ action of SUSY algebra (5).²¹ Constructing formal deformations of this formal deformation we obtain that in the reduced case all infinitesimal deformations can be extended to formal ones.

We have noticed in Section 3 that for G of the form $\delta = \nabla_i \nabla_i$ the expression $\text{tr}\theta_1 \cdots \theta_{16}G$ generates a SUSY infinitesimal deformation of unreduced YM action functional. One can prove that this deformation also can be extended to formal deformation, however, the above construction of formal deformation does not work in this case. The proof is based on the remark that infinitesimal deformation $A\text{tr}\delta$ can be applied to a formal deformation we constructed and it remains local.

²⁰ It is better to say that every infinitesimal deformation can be represented as linear combination of $\delta\mathcal{L}_{16}$ and $\text{tr}\theta_1 \cdots \theta_{16}G$.

²¹ Notice that superstring theory gives a formal SUSY deformation of SYM theory that corresponds to infinitesimal deformation represented as linear combination of $\delta\mathcal{L}_{16}$ (with non-zero coefficient) and $\text{tr}\theta_1 \cdots \theta_{16}G$. If we were able to prove that this SUSY action extends to L_∞ action we could use this deformation to extend all infinitesimal deformations in reduced case.

Appendices

A L_∞ and A_∞ algebras

Let us consider a supermanifold equipped with an odd vector field Q obeying $[Q, Q] = 0$ (a Q -manifold). Let us introduce a coordinate system in a neighborhood of a point of Q -manifold belonging to zero locus Q . Then the vector field Q considered as a derivation of the algebra of formal power series can be specified by its action on the coordinate functions z^A :

$$Q(z^A) = \sum_n \sum \pm \mu_{B_1, \dots, B_n}^A z^{B_1} \dots z^{B_n} \quad (96)$$

We can use tensors $\mu_n = \mu_{B_1, \dots, B_n}^A$ to define a series of operations. The operation μ_n has n arguments; it can be considered as a linear map $V^{\otimes n} \rightarrow V$ (here V stands for the tangent space at $x = 0$). However, it is convenient to change parity of V and consider μ_n as a symmetric map $(\Pi V)^{\otimes n} \rightarrow \Pi V$. It is convenient to add some signs in the definition of μ_n . With appropriate choice of signs we obtain that operations μ_n obey some quadratic relations; by definition the operators μ_n obeying these relations specify a structure of L_∞ algebra on $W = \Pi V$. We see that a point of zero locus of the field Q specifies an L_∞ algebra; geometrically one can say that L_∞ algebra is a formal Q -manifold. (A formal manifold is a space whose algebra of functions can be identified with the algebra of formal power series. If the algebra is equipped with odd derivation Q , such that $\{Q, Q\} = 0$ we have a structure of formal Q manifold.) The considerations of our paper are formal. This means that we can interpret all functions of fields at hand as formal power series. Therefore instead of working with Q -manifolds we can work with L_∞ algebras.

On a Q -manifold with odd symplectic structure we can choose the coordinates z^1, \dots, z^n as Darboux coordinates, i.e. we can assume that the coefficients of symplectic form do not depend on z . Then the L_∞ algebra is equipped with invariant odd inner product.

Hence we can say that L_∞ algebra specifies a classical system and L_∞ algebra with invariant odd inner product specifies a Lagrangian classical system.

It is often important to consider \mathbb{Z} -graded L_∞ -algebras (in BV-formalism this corresponds to the case when the fields are classified according to ghost number). *We assume in this case that the derivation Q raises the grading (the ghost number) by one.*

An L_∞ algebra where all operations μ_n with $n \geq 3$ vanish can be identified with differential graded Lie algebra (the operation μ_1 is the differential, μ_2 is the bracket). An L_∞ algebra corresponding to Lie algebra with zero differential is \mathbb{Z} -graded.

For L_∞ algebra $\mathfrak{g} = (W, \mu_n)$ one can define a notion of cohomology generalizing the standard notion of cohomology of Lie algebra. For example, in the case of trivial coefficients we can consider cohomology of Q acting as a derivation of the algebra $\widehat{\text{Sym}}(W^*)$ of formal functions on W (of the algebra of formal series). In the case when the L_∞ algebra corresponds to differential Lie algebra \mathfrak{g} this cohomology coincides with Lie algebra cohomology $H(\mathfrak{g}, \mathbb{C})$ (cohomology with trivial coefficients). Considering cohomology of Q acting on the space of vector fields (space of derivations of the algebra of functions) we get a notion generalizing the notion of cohomology $H(\mathfrak{g}, \mathfrak{g})$ (cohomology with coefficients in adjoint representation).²²

Notice, that to every L_∞ algebra $\mathfrak{g} = (W, \mu_n)$ we can assign a supercommutative differential algebra $(\widehat{\text{Sym}}(W^*), Q)$ that is in some sense dual to the original L_∞ -algebra. If only a finite number of operations μ_n does not vanish the operator Q transforms a polynomial function into a polynomial function, hence we can consider also a free supercommutative differential algebra $(\text{Sym}(W), Q)$ where $\text{Sym}(W)$ stands for the algebra of polynomials on W . We will use the notations $(\text{Sym}(W^*), Q) = C^\bullet(\mathfrak{g})$, $(\widehat{\text{Sym}}(W^*), Q) = \hat{C}^\bullet(\mathfrak{g})$ and the notations $H(\mathfrak{g}, \mathbb{C})$, $\hat{H}(\mathfrak{g}, \mathbb{C})$ for corresponding cohomology.²³ Similarly for the

²² Usually the definition of Lie algebra cohomology is based on the consideration of polynomial functions of ghosts; using formal series we obtain a completion of cohomology.

²³ In the case of Lie algebra the functor C^\bullet coincides with Cartan-Eilenberg construction of

cohomology in the space of derivations we use the notations $H(\mathfrak{g}, \mathfrak{g}), \hat{H}(\mathfrak{g}, \mathfrak{g})$.

In the case when L_∞ algebra is \mathbb{Z} -graded the cohomology $H(\mathfrak{g}, \mathbb{C})$ and $H(\mathfrak{g}, \mathfrak{g})$ are also \mathbb{Z} -graded.

One can consider intrinsic cohomology of an L_∞ algebra. They are defined as $\text{Ker}\mu_1/\text{Im}\mu_1$. One says that an L_∞ homomorphism, which is the same as Q -map in the language of Q -manifolds²⁴, is quasi-isomorphism if it induces an isomorphism of intrinsic cohomology. Notice, that in the case of \mathbb{Z} -graded L_∞ algebras L_∞ homomorphism should respect \mathbb{Z} grading.

Every \mathbb{Z} -graded L_∞ algebra is quasi-isomorphic to L_∞ with $\mu_1 = 0$. (In other words every L_∞ algebra has a minimal model). Moreover, every \mathbb{Z} -graded L_∞ algebra is quasi-isomorphic to direct product of minimal L_∞ algebra and a trivial one. (We say that L_∞ algebra is trivial if it can be regarded as differential abelian Lie algebra with zero cohomology.)

The role of zero locus of Q is played by the space of solutions of Maurer-Cartan (MC) equation:

$$\sum_n \frac{1}{n!} \mu_n(a, \dots, a) = 0.$$

To obtain a space of solutions Sol/\sim we should factorize space of solutions Sol of MC in appropriate way or work with a minimal model of A .

Our main interest lies in gauge theories. We consider these theories for all groups $U(n)$ at the same time. To analyze these theories it is more convenient to work with A_∞ instead of L_∞ algebras.

An A_∞ algebra can be defined as a formal non-commutative Q -manifold. In other words we consider an algebra of power series of several variables which do not satisfy any relations (some of them are even, some are odd). An A_∞ algebra is defined as an odd derivation Q of this algebra which satisfies $[Q, Q] = 0$.

More precisely we consider a \mathbb{Z}_2 -graded vector space W with coordinates z^A . The algebra of formal noncommutative power series $\mathbb{C}\langle\langle z^A \rangle\rangle$ is a completion $\hat{T}(W^*)$ of the tensor algebra $T(W^*)$ (of the algebra of formal noncommutative differential algebra giving Lie algebra cohomology).

²⁴Recall, that a map of Q -manifolds is a Q -map if it is compatible with Q .

polynomials). The derivation is specified by the action on z^A :

$$Q(z^A) = \sum_n \sum \pm \mu_{B_1, \dots, B_n}^A z^{B_1} \dots z^{B_n} \quad (97)$$

We can use μ_{B_1, \dots, B_n}^A to specify a series of operations μ_n on the space ΠW as in L_∞ case. (In the case when W is \mathbb{Z} -graded instead parity reversal Π we should consider the shift of the grading by 1.) If Q defines an A_∞ algebra then the condition $[Q, Q] = 0$ leads to quadratic relations between operations; these relations can be used to give an alternative definition of A_∞ algebra. In this definition an A_∞ algebra is a \mathbb{Z}_2 -graded or \mathbb{Z} -graded linear space, equipped with a series of maps $\mu_n : A^{\otimes n} \rightarrow A, n \geq 1$ of degree $2 - n$ that satisfy quadratic relations:

$$\sum_{i+j=n+1} \sum_{0 \leq l \leq i} \epsilon(l, j) \times \mu_i(a_0, \dots, a_{l-1}, \mu_j(a_l, \dots, a_{l+j-1}), a_{l+j}, \dots, a_n) = 0 \quad (98)$$

where $a_m \in A$, and

$$\epsilon(l, j) = (-1)^{j \sum_{0 \leq s \leq l-1} \deg(a_s) + l(j-1) + j(i-1)}.$$

In particular, $\mu_1^2 = 0$.

Notice that in the case when only finite number of operations μ_n do not vanish (the RHS of (97) is a polynomial) we can work with polynomial functions instead of power series. We obtain in this case a differential on the tensor algebra $(T(\Pi W^*), Q)$. The transition from A_∞ algebra $A = (W, \mu_n)$ to a differential algebra $\text{cobar} A = (T(\Pi W^*), Q)$ is known as co-bar construction. If we consider instead of tensor algebra its completion (the algebra of formal power series) we obtain the differential algebra $(\hat{T}(\Pi W^*), Q)$ as a completed co-bar construction $\widehat{\text{cobar} A}$.

The cohomology of differential algebra $(T(\Pi W^*), Q) = \text{cobar} A$ are called Hochschild cohomology of A with coefficients in trivial module \mathbb{C} ; they are denoted by $HH(A, \mathbb{C})$. Using the completed co-bar construction we can give another definition of Hochschild cohomology of A_∞ algebra as the cohomology of the differential algebra $(\hat{T}(\Pi W^*), Q) = \widehat{\text{cobar} A}$; this cohomology can be defined also in

the case when we have infinite number of operations. It will be denoted by $\widehat{HH}(A, \mathbb{C})$. Under some mild conditions (for example, if the differential is equal to zero) one can prove that $\widehat{HH}(A, \mathbb{C})$ is a completion of $HH(A, \mathbb{C})$; in the case when $HH(A, \mathbb{C})$ is finite-dimensional this means that the definitions coincide. We will always assume that $\widehat{HH}(A, \mathbb{C})$ is a completion of $HH(A, \mathbb{C})$.

The theory of A_∞ algebras is very similar to the theory of L_∞ algebras. In particular μ_1 is a differential: $\mu_1^2 = 0$. It can be used to define intrinsic cohomology of A_∞ algebra. If $\mu_n = 0$ for $n \geq 3$ then operations μ_1, μ_2 define a structure of differential associative algebra on W .

The role of equations of motion is played by so called MC equation

$$\sum_{n \geq 1} \mu_n(a, \dots, a) = 0 \quad (99)$$

Again to get a space of solutions Sol / \sim we should factorize solutions of MC equation in appropriate way or to work in a framework of minimal models, i.e. we should use the A_∞ algebra that is quasi-isomorphic to the original algebra and has $\mu_1 = 0$. (Every \mathbb{Z} -graded A_∞ algebra has a minimal model.)

We say that 1 is the unit element of A_∞ algebra if $\mu_2(1, a) = \mu_2(a, 1) = a$ (i.e. 1 is the unit for binary operation) and all other operations with 1 as one of arguments give zero. For every A_∞ algebra A we construct a new A_∞ algebra \tilde{A} adjoining a unit element.²⁵

Having an A_∞ algebra A we can construct a series $L_N(A)$ of L_∞ algebras. If $N = 1$ it is easy to describe the corresponding L_∞ algebra in geometric language. There is a map from noncommutative formal functions on ΠA to ordinary (super)commutative formal functions on the same space. Algebraically it corresponds to imposing (super) commutativity relations among generators. Derivation Q is compatible with such modification. It results in $L_1(A)$. By definition $L_N(A) = L_1(A \otimes \text{Mat}_N)$.

²⁵ Notice, that in our definition of Hochschild cohomology we should work with non-unital algebras; otherwise the result for the cohomology with coefficients in \mathbb{C} would be trivial. In more standard approach one defines Hochschild cohomology of unital algebra using the augmentation ideal.

If A is an ordinary associative algebra, then $L_1(A)$ is in fact a Lie algebra- it has the same space and the operation is equal to the commutator $[a, b] = ab - ba$.

The use of A_∞ algebras in the YM theory is based on the remark that one can construct an A_∞ algebra \mathcal{A} with inner product such that for every N the algebra $L_N(\tilde{\mathcal{A}})$ specifies YM theory with matrices of size $N \times N$ in BV formalism. (Recall, that we construct $\tilde{\mathcal{A}}$ adjoining unit element to \mathcal{A} .) The construction of the A_∞ algebra $\tilde{\mathcal{A}}$ is very simple: in the formula for Q in BV-formalism of YM theory in component formalism we replace matrices with free variables. The operator Q obtained in this way specifies also differential algebras $\text{cobar}\tilde{\mathcal{A}}$ and $\widehat{\text{cobar}\tilde{\mathcal{A}}}$. To construct the A_∞ algebra \mathcal{A} in the case of reduced YM theory we notice that the elements of the basis of $\tilde{\mathcal{A}}$ correspond to the fields of the theory; the element corresponding to the ghost field c is the unit; remaining elements of the basis generate the algebra \mathcal{A} . In the case of reduced theory the differential algebra $\text{cobar}\mathcal{A}$ can be obtained from $\text{cobar}\tilde{\mathcal{A}}$ by means of factorization with respect to the ghost field c ; we denote this algebra by BV_0 and the original algebra \mathcal{A} will be denoted by bv_0 . The construction in unreduced case is similar. In this case the ghost field (as all other fields) is a function on ten-dimensional space; to obtain $\text{cobar}\mathcal{A}$ (that will be denoted later by BV) we factorize $\text{cobar}\tilde{\mathcal{A}}$ with respect to the ideal generated by the constant ghost field c . We will use the notation bv for the algebra \mathcal{A} in unreduced case.

Instead of working with component fields we can use pure spinors. Then instead of the algebra bv_0 we should work with reduced Berkovits algebra B_0 that is quasi-isomorphic to bv_0 ; the algebra BV_0 is quasi-isomorphic to $U(YM)$. In unreduced case we work with Berkovits algebra B that is quasi-isomorphic to bv and with the algebra $U(TYM)$ quasi-isomorphic to BV (see Section 6, [18] and [19] for more detail).

Notice, that the quasi-isomorphisms we have described are useful for calculation of homology. For example, as we have seen in Section 6 the space of fields in pure spinor formalism can be equipped with odd symplectic form (86) that vanishes if the sum of ghost numbers of arguments is > 3 ; the space of fields should be factorized with respect to the kernel of this form. It follows that

homology and cohomology of $U(YM)$ with coefficients in any module vanish in dimensions > 3 . From the other side the form (86) can be used to establish Poincaré duality in the cohomology of $U(YM)$.

It is easy to reduce classification of deformations of A_∞ algebra A to a homological problem (see [22]). Namely it is clear that an infinitesimal deformation of Q obeying $[Q, Q] = 0$ is an odd derivation q obeying $[Q, q] = 0$. The operator Q specifies a differential on the space of all derivations by the formula

$$\tilde{Q}q = [Q, q] \quad (100)$$

We see that infinitesimal deformations correspond to cocycles of this differential. It is easy to see that two infinitesimal deformations belonging to the same cohomology class are equivalent (if $q = [Q, v]$ where v is a derivation then we can eliminate q by a change of variables $exp(v)$). We see that the classes of infinitesimal deformations can be identified with homology $H(Vect(\mathbb{V}), d)$ of the space of vector fields. (Vector fields on \mathbb{V} are even and odd derivations of \mathbb{Z}_2 -graded algebra of formal power series.) If the number of operations is finite we can restrict ourselves to polynomial vector fields (in other words, we can replace $Vect(\mathbb{V})$ with $cobar A \otimes A$).

The above construction is another particular case of Hochschild cohomology (the cohomology with coefficients in coefficients in \mathbb{C} was defined in terms of cobar construction.) We denote it by $\widehat{HH}(A, A)$ (if we are working with formal power series) or by $HH(A, A)$ (if we are working with polynomials). Notice that these cohomologies have a structure of (super) Lie algebra induced by commutator of vector fields.

We will give a definition of Hochschild cohomology of differential graded associative algebra (A, d_A)

$$A = \bigoplus_{i \geq 0} A_i$$

with coefficients in a differential bimodule (M, d_M)

$$M = \bigoplus_i M_i$$

in terms of Hochschild cochains (multilinear functionals on A with values in M).

We use the standard notation for the degree $\bar{a} = i$ of a homogeneous element $a \in A_i$.

We first associate with the pair (A, M) a bicomplex $(C^{n,m}, D_I, D_{II}), n \geq 0$, $D_I : C^{n,m} \rightarrow C^{n+1,m}, D_{II} : C^{n,m} \rightarrow C^{n,m+1}$ as follows:

$$C^{n,m}(A, M) = \prod_{i_1, \dots, i_n} \text{Hom}(A_{i_1} \otimes \dots \otimes A_{i_n}, M_{m+i_1+\dots+i_n}) \quad (101)$$

and for $c \in C^{n,m}$

$$\begin{aligned} D_I c &= a_0 c(a_1, \dots, a_n) + \sum_{i=0}^{n-1} (-1)^{i+1} c(a_0, \dots, a_i a_{i+1}, \dots, a_n) + (-1)^{m\bar{a}_n+n} c(a_0, \dots, a_{n-1}) a_n \\ D_{II} c &= \sum_{i=1}^n (-1)^{1+\bar{a}_1+\dots+\bar{a}_{i-1}} c(a_1, \dots, d_A(a_i), \dots, a_n) + (-1)^k d_M c(a_1, \dots, a_n) \end{aligned} \quad (102)$$

Clearly

$$D_I^2 = 0, D_{II}^2 = 0, D_I D_{II} + D_{II} D_I = 0$$

We define the space of Hochschild i -th cochains as

$$\widehat{C}^i(A, M) = \prod_{n+m=i} C^{n,m}(A, M). \quad (103)$$

Then $\widehat{C}^\bullet(A, M)$ is the complex $(\prod C^i(A, M), D)$ with $D = D_I + D_{II}$. The operator D can also be considered as a differential on the direct sum $C(A, M) = \bigoplus_i C^i(A, M)$ with direct products in (101,103) replaced by the direct sums (on the space of non-commutative polynomials on ΠA with values in M). Similarly $\widehat{C}(A, M)$ can be interpreted as the space of formal power series on A with values in M . We define the Hochschild cohomology $HH(A, M)$ and $\widehat{HH}(A, M)$ as the cohomology of this differential. Again under certain mild conditions that will be assumed in our consideration the second group is a completion of the first one; the groups coincide if $HH(A, M)$ is finite-dimensional.

Notice that $C(A, M)$ can be identified with the tensor product $\text{cobar} A \otimes M$ with a differential defined by the formula

$$D(c \otimes m) = (d_{\text{cobar}} + d_M)c \otimes m + [e, c \otimes m] \quad (104)$$

where e is the tensor of the identity map $\text{id} \in \text{End}(A) \cong \Pi A^* \otimes A \subset \text{cobar}(A) \otimes A$

A similar statement is true for $\hat{C}(A, M)$.

Notice that we can define the total grading of Hochschild cohomology $HH^i(A, M)$ where i stands for the total grading defined in terms of A , M and the ghost number (the number of arguments).

In the case when M is the algebra A considered as a bimodule the elements of $HH^2(A, A)$ label infinitesimal deformations of associative algebra A and the elements of $HH^\bullet(A, A)$ label infinitesimal deformations of A into A_∞ algebra. Derivations of A specify elements of $HH^1(A, A)$ (more precisely, a derivation can be considered as one-dimensional Hochschild cocycle; inner derivations are homologous to zero).

We can define Hochschild homology HH_\bullet considering Hochschild chains (elements of $A \otimes \cdots \otimes A \otimes M$). If A and M are finite-dimensional (or graded with finite-dimensional components) we can define homology by means of dualization of cohomology

$$HH_i(A, M) = HH^i(A, M^*)^*.$$

Let us assume that the differential bimodule M is equipped with bilinear inner product of degree n ²⁶ that descends to non-degenerate inner product on homology. This product generates a quasi-isomorphism $M \rightarrow M^*$ and therefore an isomorphism between $HH_i(A, M)$ and $HH^{n-i}(A, M)$ (Poincaré isomorphism).

Let us suppose now that A_∞ has Lie algebra of symmetries \mathfrak{g} and we are interested in deformations of this algebra preserving the symmetries. This problem appears if we consider YM theory for all groups $U(n)$ at the same time and we would like to deform the equations of motion preserving the symmetries of the original theory (however we do not require that the deformed equations come from an action functional).

When we are talking about symmetries of A_∞ algebra A we have in mind derivations of the algebra $\widehat{\text{cobar}} A = (\hat{T}(W^*), Q)$ (vector fields on a formal non-

²⁶ This means that the inner product does not vanish only if the sum of degrees of arguments is equal to n . For example, the odd bilinear form in pure spinor formalism of SYM can be considered as inner product of degree 3.

commutative manifold) that commute with Q ; see equation (97). We say that symmetries q_1, \dots, q_k form Lie algebra \mathfrak{g} if they satisfy commutation relations of \mathfrak{g} up to Q -exact terms. These symmetries determine a homomorphism of Lie algebra \mathfrak{g} into Lie algebra $\widehat{HH}(A, A)$. We will say that this homomorphism specifies weak action of \mathfrak{g} on A .

In the case when A_∞ algebra is \mathbb{Z} -graded we can impose the condition that the symmetry is compatible with the grading.

Another way to define symmetries of A_∞ algebra is to identify them with L_∞ actions of Lie algebra \mathfrak{g} on this algebra, i.e. with L_∞ homomorphisms of \mathfrak{g} into differential Lie algebra of derivations $Vect$ of the algebra $\widehat{\text{cobar}}A$ (the differential acts on $Vect$ as (super)commutator with Q). More explicitly L_∞ action is defined as a linear map

$$q : \text{Sym}\Pi\mathfrak{g} \rightarrow \Pi Vect \quad (105)$$

or as an element of odd degree

$$q \in C^\bullet(\mathfrak{g}) \otimes Vect \quad (106)$$

obeying

$$d_{\mathfrak{g}}q + [Q, q] + \frac{1}{2}[q, q] = 0. \quad (107)$$

where $d_{\mathfrak{g}}$ is a differential entering the definition of Lie algebra cohomology. We can write q in the form

$$q = \sum \frac{1}{r!} q_{\alpha_1, \dots, \alpha_r} c^{\alpha_1} \dots c^{\alpha_r}$$

where c^α are ghosts of the Lie algebra; here $d_{\mathfrak{g}} = \frac{1}{2} f_{\beta\gamma}^\alpha c^\beta c^\gamma \frac{\partial}{\partial c^\alpha}$ where $f_{\alpha\beta}^\gamma$ denote structure constants of \mathfrak{g} .

One can represent (107) as an infinite sequence of equations for the coefficients; the first of these equations has the form

$$[q_\alpha, q_\beta] = f_{\alpha\beta}^\gamma q_\gamma + [Q, q_{\alpha\beta}].$$

We see that q_α satisfy commutation relations of \mathfrak{g} up to Q -exact terms (as we have said this means that they specify a weak action of \mathfrak{g} on A and a homomorphism $\mathfrak{g} \rightarrow \widehat{HH}(A, A)$).

In the remaining part of this section we use the notation HH instead of \widehat{HH} .

Let us consider now an A_∞ algebra A equipped with L_∞ action of Lie algebra \mathfrak{g} . To describe infinitesimal deformations of A preserving the Lie algebra of symmetries we should find solutions of equations (107) and $[Q, Q] = 0$ where Q is replaced by $Q + \delta Q$ and q by $q + \delta q$. After appropriate identifications these solutions can be described by elements of cohomology group that will be denoted by $HH_{\mathfrak{g}}(A, A)$. To define this group we introduce ghosts c^α . In other words we multiply $Vect(\mathbb{V})$ by $\Lambda(\Pi\mathfrak{g}^*)$ and define the differential by the formula

$$d = \tilde{Q} + \frac{1}{2} f_{\beta\gamma}^\alpha c^\beta c^\gamma \frac{\partial}{\partial c^\alpha} + q_\alpha c^\alpha + \dots \quad (108)$$

The dots denote the terms having higher order with respect to c^α . They should be included to satisfy $d^2 = 0$ if q_α obey commutation relations of \mathfrak{g} up to Q -exact term. They can be expressed in terms of $q_{\alpha_1, \dots, \alpha_r}$:

$$d = \tilde{Q} + \frac{1}{2} f_{\beta\gamma}^\alpha c^\beta c^\gamma \frac{\partial}{\partial c^\alpha} + \sum_{r \geq 1} \frac{1}{r!} c^{\alpha_1} \dots c^{\alpha_r} q_{\alpha_1, \dots, \alpha_r} \quad (109)$$

In the terminology introduced in Section 6 $HH_{\mathfrak{g}}(A, A)$ is the Lie algebra cohomology of \mathfrak{g} with coefficients in the L_∞ differential \mathfrak{g} -module $(Vect(\mathbb{V}), \tilde{Q})$:

$$HH_{\mathfrak{g}}(A, A) = H(\mathfrak{g}, (Vect(\mathbb{V}), \tilde{Q})). \quad (110)$$

From other side in the case of trivial \mathfrak{g} we obtain Hochschild cohomology. Therefore we will use the term Lie- Hochschild cohomology for the group (110).

Every deformation of A_∞ algebra A induces a deformation of the algebra \tilde{A} and of the corresponding L_∞ algebra $L_N(\tilde{A})$; if A_∞ algebra has Lie algebra of symmetries \mathfrak{g} then the same is true for this L_∞ algebra. Deformations of A_∞ algebra preserving the symmetry algebra \mathfrak{g} induce symmetry preserving deformations of the L_∞ algebra²⁷. This remark permits us to say that the calculations of symmetry preserving deformations of A_∞ algebra A corresponding to

²⁷At the level of cohomology groups it means that we have a series of homomorphisms $HH_{\mathfrak{g}}(A, A) \rightarrow H_{\mathfrak{g}}(L_N(\tilde{A}), L_n(\tilde{A}))$

YM theory induces symmetry preserving deformations of EM for YM theories with gauge group $U(N)$ for all N .

The calculation of cohomology groups $HH_{\text{SUSY}}(YM, YM)$ permits us to describe SUSY-invariant deformations of EM. However we would like also to characterize Lagrangian deformations of EM. This problem also can be formulated in terms of homology. Namely we should consider A_∞ algebras with invariant inner product and their deformations. We say that A_∞ algebra A is equipped with odd invariant nondegenerate inner product $\langle \cdot, \cdot \rangle$ if $\langle a_0, \mu_n(a_1, \dots, a_n) \rangle = (-1)^{n+1} \langle a_n, \mu_n(a_0, \dots, a_{n-1}) \rangle$. It is obvious that the corresponding L_∞ algebras $L_N(A)$ are equipped with odd invariant inner product. Therefore the corresponding vector field Q comes from a solution of a Master equation $\{S, S\} = 0$ (i.e. we have Lagrangian equations of motion). We will check that the deformations of A_∞ algebra preserving invariant inner product are labeled by cyclic cohomology of the algebra [22].

As we have seen the deformations of A_∞ algebra are labeled by Hochschild cohomology cocycles of differential \tilde{Q} (see formula (100)) acting on the space of derivations $Vect(\mathbb{V})$.

A derivation ρ is uniquely defined by its values on generators of the basis of vector space W^* (on generators of algebra $\hat{T}(W^*)$). Let us introduce notations $\rho(z^i) = \rho^i(z^1, \dots, z^n)$. The condition that ρ specifies a cocycle of d means that it specifies a Hochschild cocycle with coefficients in A . The condition that ρ preserves the invariant inner product is equivalent the cyclicity condition on $\rho_{i_0, i_1 \dots i_n}$, where $\rho_{i_0}(z^1, \dots, z^n) = \sum \rho_{i_0, i_1 \dots i_k} z^{i_1} \dots z^{i_k}$. (We lower the upper index in ρ using the invariant inner product.) The cyclicity condition has the form

$$\rho_{i_0, i_1 \dots i_k} = (-1)^{k+1} \rho_{i_k, i_0 \dots i_{k-1}} \quad (111)$$

We say that $\rho_{i_0, i_1 \dots i_k}$ obeying formula (111) is a cyclic cochain. To define cyclic cohomology we use Hochschild differential on the space of cyclic cochains.

28

²⁸One can say that the vector field ρ preserving inner product is a Hamiltonian vector field. The cyclic cochain $\rho_{i_0, i_1 \dots i_k}$ can be considered as its Hamiltonian. The differential (100) acts

If we consider deformations of A_∞ algebra with inner product and Lie algebra \mathfrak{g} of symmetries and we are interested in deformations of A to an algebra that also has invariant inner product and the same algebra of symmetries we should consider cyclic cohomology $HC_{\mathfrak{g}}(A)$. The definition of this cohomology can be obtained if we modify the definition of $HC(A)$ in the same way as we modified the definition of $HH(A, A)$ to $HH_{\mathfrak{g}}(A, A)$.

It is obvious that there exist a homomorphism from $HC(A)$ to $HH(A, A)$ and from $HC_{\mathfrak{g}}(A)$ to $HH_{\mathfrak{g}}(A, A)$ (every deformation preserving inner product is a deformation).²⁹ Our main goal is to calculate the image of $HC_{\mathfrak{g}}(A)$ in $HH_{\mathfrak{g}}(A, A)$ for the case of A_∞ algebra of YM theory, i.e. we would like to describe all supersymmetric deformations of YM that come from a Lagrangian.

Cyclic cohomology are related to Hochschild cohomology by Connes exact sequence:

$$\dots \rightarrow HC^n(A) \rightarrow HH^n(A, A^*) \rightarrow HC^{n-1}(A) \rightarrow HC^{n+1}(A) \rightarrow \dots$$

Similar sequence exists for Lie-cyclic cohomology.

To define the cyclic homology $HC_{\bullet}(A)$ we work with cyclic chains (elements of $A \otimes \dots \otimes A$ factorized with respect to the action of cyclic group). The natural map of Hochschild chains with coefficients in A to cyclic chains commutes with the differential and therefore specifies a homomorphism $HH_k(A, A) \xrightarrow{I} HC_k(A)$. This homomorphism enters the homological version of Connes exact sequence

$$\dots \rightarrow HC_{n-1}(A) \xrightarrow{b} HH_n(A, A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \rightarrow \dots$$

We define the differential $B : HH_n(A, A) \rightarrow HH_{n+1}(A, A)$ (Connes differential) as a composition $b \circ I$.

on the space of Hamiltonian vector fields. The cohomology of corresponding differential acting on the space of Hamiltonians is called cyclic cohomology.

²⁹Notice that we have assumed that A is equipped with non-degenerate inner product. The definition of cyclic cohomology does not require the choice of inner product; in general there exists a homomorphism $HC(A) \rightarrow HH(A, A^*)$. The homomorphism $HC(A) \rightarrow HH(A, A)$ can be obtained as a composition of this homomorphism with a homomorphism $HH(A, A^*) \rightarrow HH(A, A)$ induced by a map $A^* \rightarrow A$.

An interesting refinement of Connes exact sequence exists in the case when A is the universal enveloping of a Lie algebra \mathfrak{g} over \mathbb{C} . In this case cyclic homology get an additional index: $HC_{k,j}(A)$. Such groups fit into the long exact sequence [13]:

$$\cdots \rightarrow HC_{n-1,i}(U(\mathfrak{g})) \xrightarrow{b_{n-1,i}} HH_n(U(\mathfrak{g}), \text{Sym}^i(\mathfrak{g})) \xrightarrow{I_{n,i}} HC_{n,i+1}(U(\mathfrak{g})) \xrightarrow{S_{n,i+1}} HC_{n-2,i}(U(\mathfrak{g})) \rightarrow \cdots$$

The differential

$$B_i : HH_n(U(\mathfrak{g}), \text{Sym}^i(\mathfrak{g})) \rightarrow HH_{n+1}(U(\mathfrak{g}), \text{Sym}^{i-1}(\mathfrak{g}))$$

is defined as a composition $b_{n,i+1} \circ I_{n,i}$. Finally if the Lie algebra \mathfrak{g} is graded then all homological constructs acquire an additional bold index: $HH_{nl}(U(\mathfrak{g}), \text{Sym}^i(\mathfrak{g}))$, $HC_{n,i,l}(U(\mathfrak{g}))$. This index is preserved by the differential in the above sequence.

It is worthwhile to mention that all natural constructions that exist in cyclic homology can be extended to Lie-cyclic homology.

It is important to emphasize that homology and cohomology theories we considered in this section are invariant with respect to quasi-isomorphism (under certain conditions that are fulfilled in our situation).³⁰

According to [12] a quasi-isomorphism of two algebras $A \rightarrow B$ induces an isomorphism in Hochschild cohomology $HH^\bullet(A, A) \cong HH^\bullet(B, B)$. As we have mentioned Hochschild cohomology $HH^\bullet(A, A)$ is equipped with a structure of super Lie algebra, the isomorphism is compatible with this structure.

This theorem guarantees that quasi-isomorphism $A \rightarrow B$ allows us to translate a weak \mathfrak{g} action from A to B .

We have defined L_∞ action as an L_∞ homomorphism of Lie algebra \mathfrak{g} into differential Lie algebra of derivations $Vect(A)$. It follows from the results of [12] that a quasi-isomorphism $\phi : A \rightarrow B$ induces a quasi-isomorphism $\tilde{\phi} : Vect(A) \rightarrow Vect(B)$ compatible with L_∞ structure.³¹ We obtain that L_∞ action on A can be transferred to an L_∞ action on quasi-isomorphic algebra B .

³⁰The most general results and precise formulation of this statement can be found in [12] for Hochschild cohomology and in [11] for cyclic cohomology.

³¹In fact the structure of $Vect(A)$ is richer: it is a B_∞ algebra (see [12] for details), but we

The calculation of cohomology groups we are interested in is a difficult problem. To solve this problem we apply the notion of duality of associative and A_∞ algebras.

B Duality

We define a pairing of two differential graded augmented ³² algebras A and B as a degree one element $e \in A \otimes B$ that satisfies Maurer-Cartan equation

$$(d_A + d_B)e + e^2 = 0 \quad (112)$$

Here we understand $A \otimes B$ as a completed tensor product.

Example 27 *Let x_1, \dots, x_n be the generating set of the quadratic algebra A . The set ξ^1, \dots, ξ^n generates the dual quadratic algebra $A^!$ (see preliminaries). The element $e = x_i \otimes \xi^i$ has degree one, provided x_i and ξ^i have degrees two and minus one. The element e satisfies $e^2 = 0$ - a particular case of (112) for algebras with zero differential and therefore specifies a pairing between A and $A^!$.*

Notice that the grading we are using here differs from the grading in the Section 2.

Remark. Many details of the theory depend on the completion of the tensor product, mentioned in the definition of e . We, however, chose to ignore this issue because the known systematic way to deal with it requires introduction of a somewhat artificial language of co-algebras. ³³

We call a non-negatively (non-positively) graded differential algebra $A = \bigoplus_i A_i$ connected, if $A_0 \cong \mathbb{C}$. Such algebra is automatically augmented $\epsilon : A \rightarrow$

will use only L_∞ (Lie) structure. One of the results of [12] asserts that $\tilde{\phi}$ is compatible with B_∞ structure. As a corollary it induces a quasi-isomorphism of L_∞ structures.

³²A differential graded algebra A is called augmented if it is equipped with a d -invariant homomorphism $\epsilon : A \rightarrow \mathbb{C}$ of degree zero. We assume that the algebras at hand are \mathbb{Z} -graded and graded components are finite-dimensional.

³³One can define the notion of duality between algebra and co-algebra. This notion has better properties than the duality between algebras.

A_0 . We call a non-negatively graded connected algebra A simply-connected if $A_1 \cong 0$.

Let us consider a differential graded algebra $\text{cobar}A = (T(\Pi A^*), d)$ where A is an associative algebra and d is the Hochschild differential. In other words we consider the co-bar construction for the algebra A .

Proposition 28 *The pairing e defines the map*

$$\rho : \text{cobar}(A) \rightarrow B$$

of differential graded algebras.

Proof. The algebra $\text{cobar}(A)$ is generated by elements of ΠA^* . The value of the map ρ on $l \in \Pi A^*$ is equal to

$$\rho(l) = \langle l, f_i \rangle g^i$$

where $e = f_i \otimes g^i \in A \otimes B$. The compatibility of ρ with the differential follows automatically from (112). (Notice, that for graded spaces we always consider the dual as graded dual, i.e. as a direct sum of dual spaces to the graded components.)

■

Similarly the element e defines a map $\text{cobar}(B) \rightarrow A$.

Definition 29 *The differential algebras A and B are dual if there exists a pairing (A, B, e) such that the maps $\text{cobar}(A) \rightarrow B$ and $\text{cobar}(B) \rightarrow A$ are quasi-isomorphisms.*

34

Notice that duality is invariant with respect to quasi-isomorphism.

If A is quadratic then A is dual to $A^!$ iff A is a Koszul algebra.

If a differential graded algebra A has a dual algebra, then A is dual to $\text{cobar}A$. If A is a connected and simply-connected differential graded algebra, i.e. $A = \bigoplus_{i \geq 0} A_i$ and $A_0 = \mathbb{C}$ and $A_1 = 0$, then A and $\text{cobar}A$ are dual.

³⁴Very similar notion of duality was suggested independently by Kontsevich [10].

If differential graded algebras A and B are dual it is clear that Hochschild cohomology $HH(A, \mathbb{C})$ of A with trivial coefficients coincides with intrinsic cohomology of B . This is because B is quasi-isomorphic to $\text{cobar}(A)$. One say also that

$$HH(A, A) = HH(B, B), \quad (113)$$

This is clear because these cohomology can be calculated in terms of complex $A \otimes B$, that is quasi-isomorphic both to $A \otimes \text{cobar}A$ and $\text{cobar}B \otimes B$.

This statement can be generalised to Hochschild cohomology of A with coefficients in any bimodule M . Namely, we should introduce in $B \otimes M$ a differential by the formula

$$d(b \otimes m) = (d_B + d_M)b \otimes m + [e, b \otimes m] \quad (114)$$

Proposition 30 *Let A be a connected and simply-connected differential graded algebra, i.e. $A = \bigoplus_{i \geq 0} A_i$ and $A_0 = \mathbb{C}$ and $A_1 = 0$. Then the Hochschild cohomology $HH(A, M)$ coincide with the cohomology of $B \otimes M$ with respect to differential (114).*

To prove this statement we notice that the quasi-isomorphism $\text{cobar}A \rightarrow B$ induces a homomorphism $C(A, M) = \text{cobar}A \otimes M \rightarrow B \otimes M$; it follows from (104) that this homomorphism commutes with the differentials and therefore induces a homomorphism on homology. The induced homomorphism is an isomorphism; this can be derived from the fact that the map $\text{cobar}A \rightarrow B$ is a quasi-isomorphism. (The derivation is based on the techniques of spectral sequences; the condition on algebra A guarantees the convergence of spectral sequence.)

The above proposition can be applied to the case when A is a Koszul quadratic algebra and $B = A^!$ is the dual quadratic algebra. We obtain the Proposition 4, stated in Section 2.2.

Proposition 31 *If differential graded algebra A is dual to B and quasi-isomorphic*

to the envelope $U(\mathfrak{g})$ of Lie algebra \mathfrak{g} then B is quasi-isomorphic to the super-commutative differential algebra $C^\bullet(\mathfrak{g})$.

This statement follows from the fact that the cohomology of $C^\bullet(\mathfrak{g})$ (=Lie algebra cohomology of \mathfrak{g}) coincides with Hochschild cohomology of $U(\mathfrak{g})$ with trivial coefficients.

It turns out that it is possible to calculate cyclic and Hochschild cohomology of A in terms of suitable homological constructions for a dual algebra B .

Let A and B be dual differential graded algebras. Let us assume that A and B satisfy assumptions of Proposition 30.

Proposition 32 *Under above assumptions there is a canonical isomorphism*

$$HC_{-1-n}(A) \cong HC^n(B),$$

where $HC^n(HC_n)$ stands for i -th cohomology (resp. homology) of an algebra.

Proposition 33 *Under the above assumptions there is an isomorphism*

$$HH^n(A, A^*) = HH_{-n}(B, B),$$

where $HH^n(HH_n)$ stands for n -th Hochschild cohomology (resp. homology).

For the case when A and B are quadratic algebras these two propositions were proven in [9]. The proof in general case is similar. It can be based on results of [6] or [13].

Let us illustrate some of above theorems on concrete examples. The algebra \mathcal{S} is dual to $U(L)$. This means, that

$$HH^\bullet(\mathcal{S}, \mathcal{S}) = HH^\bullet(U(L), U(L)) = H^\bullet(L, U(L)).$$

The reduced Berkovits algebra B_0 is dual to $U(YM)$, hence

$$HH^\bullet(B_0, B_0) = HH^\bullet(U(YM), U(YM)) = H^\bullet(YM, U(YM))$$

.

We need the following information about this cohomology ([20]):

$$\begin{aligned} H^0(YM, U(YM)) &\cong \mathbb{C} \\ H^1(YM, U(YM)) &\cong \mathbb{C} + S^* + \Lambda^2(V) + V + S^* \end{aligned} \tag{115}$$

Notice, that the answer for $H^1(YM, U(YM)) = HH^1(U(YM), U(YM))$ has clear physical interpretation: symmetries of SYM theory (translations, Lorenz transformations and supersymmetries) specify derivations of the algebra $U(YM)$.

Representing $U(YM)$ as $\text{Sym}(YM)$ we obtain additional grading on cohomology:

$$\begin{aligned} H^1(YM, \text{Sym}^1(YM)) &\cong \mathbb{C} + S^* + \Lambda^2(V) \\ H^1(YM, \text{Sym}^0(YM)) &\cong V + S^* \\ H^1(YM, \text{Sym}^i(YM)) &= 0, i \geq 2 \end{aligned} \tag{116}$$

It follows from the remarks in Appendix A that $H^k(YM, U(YM)) = HH^k(U(YM), U(YM)) = 0$ for $k > 3$.

As we mentioned in Section 6 the odd symplectic structure in pure spinor formalism is specified by degenerate closed two-form ω . This form determines an odd inner product of degree 3 on B_0 that generates Poincaré isomorphism $H^i(YM, U(YM)) \stackrel{P}{\cong} H_{3-i}(YM, U(YM))$.

C On the relation of the Lie algebra and BV approaches to deformation problem

Our main goal is to calculate SUSY deformations of 10D YM theory and its reduction to a point. In Section 4 we have reduced this question to a homological problem. Another reduction of this kind comes from BV formalism (Section 6 and Appendix A). Here we will relate these two approaches. For simplicity we will talk mostly about the reduced case; we will describe briefly the modifications that are necessary in the unreduced case.

We will use the fact that under certain conditions all objects we are interested in are invariant with respect to quasi- isomorphisms.

We can study the symmetries of Yang-Mills theory using the A_∞ algebra \mathcal{A} constructed in Appendix A or any other algebra that is quasi-isomorphic to \mathcal{A} . In BV formalism a Lie algebra action should be replaced by weak action or by L_∞ action. It will be important for us to work with L_∞ action, because this action is used in the construction of formal deformations (Section 7). We consider the case of YM theory dimensionally reduced to a point ; in this case we use the notation $\mathcal{A} = bv_0$ and the algebra $\text{cobar}bv_0 = BV_0$ is quasi-isomorphic to $U(YM)$ (to the envelope of Lie algebra YM); see [19], Theorem 1. ³⁵

The algebra bv_0 is dual to the algebra BV_0 . This means that bv_0 is quasi-isomorphic to $C^\bullet(YM)$ (to the differential commutative algebra that computes Lie algebra cohomology with trivial coefficients; see Appendix B, (31)).

One can construct an L_∞ action of the reduced supersymmetry algebra $\mathfrak{g} = \Pi\mathbb{C}^{16}$ on the algebras $bv_0, C^\bullet(YM), U(YM)$. It is sufficient to construct such an action on one of these algebras.

Let us describe the action on $C^\bullet(YM)$.

We will use the Lie algebra L (12). By construction L as a linear space is isomorphic to the direct sum $S + YM$, where $S = L^1$ is spanned by $\theta_1, \dots, \theta_{16}$. Thus

$$C^\bullet(L) \cong \mathbb{C}[[t^1, \dots, t^{16}]] \otimes C^\bullet(YM),$$

where t^α are even variables dual to θ_α . The differential d_L in $C^\bullet(L)$ is the sum

$$d_L = d_{YM} + q,$$

where q is equal to $t^\alpha t^\beta q_{\alpha\beta} + t^\gamma q_\gamma$. The operators $q_{\alpha\beta}, q_\gamma$ are derivations of $C^\bullet(YM)$. We can interpret q as map of $\text{Sym}(\Pi\mathfrak{g}) = \text{Sym}(\mathbb{C}^{16})$ into the space of derivations of $C^\bullet(YM)$. It is easy to check, that this map obeys (107); hence it specifies L_∞ action of $\mathfrak{g} = \Pi\mathbb{C}^{16}$ on $C^\bullet(YM)$.

³⁵The algebra BV_0 has as generators the generators of $U(YM)$, corresponding antifields and c^* (the antifield for ghost) ; sending antifields and c^* to zero we obtain a homomorphism of differential algebras. (Recall that the differential on $U(YM)$ is trivial.) It has been proven in [19] that this homomorphism is a quasi-isomorphism.

Another way to describe this L_∞ action of $\mathfrak{g} = \Pi\mathbb{C}^{16}$ is to construct the corrections that arise because the dimensionally reduced supersymmetries q_γ defined by the formula $q_\gamma = [\theta, x]$ anti-commute only on-shell. The operators $q_{\alpha\beta}$ can be interpreted as L_∞ corrections to the action of the Lie algebra $\mathfrak{g} = \Pi\mathbb{C}^{16}$. In this construction no higher order operators $q_{\alpha_1, \dots, \alpha_n}$ ($n \geq 3$) are present.

We should say a word of caution. The action of $\Pi\mathbb{C}^{16}$ on bv_0 constructed this way could be incompatible with the inner product. A refined version of this action, free from the shortcoming, is constructed in Appendix D.

Similar arguments permit us to construct an L_∞ action of SUSY Lie algebra in unreduced case. In this case the algebra \mathcal{A} is denoted by bv , $\text{cobar}bv = BV$ is quasi-isomorphic to $U(TYM)$ and bv is quasi-isomorphic to $C^\bullet(TYM)$. To construct an L_∞ action of SUSY Lie algebra on $C^\bullet(TYM)$ we notice that as a vector space L can be represented as a direct sum of vector subspaces $L^1 + L^2$ and TYM . This means that

$$C^\bullet(L) \cong \mathbb{C}[[t^1, \dots, t^{16}]] \otimes \Lambda[\xi^1, \dots, \xi^{10}] \otimes C^\bullet(TYM),$$

where t^α, ξ^i can be interpreted as even and odd ghosts of the Lie algebra $\mathfrak{su}(5)\mathfrak{h}$. Again we can construct the L_∞ $\mathfrak{su}(5)\mathfrak{h}$ action on $C^\bullet(TYM)$ using the differential d_L acting on $C^\bullet(L)$. Namely, we choose a basis $\langle e_\gamma \rangle, \gamma \geq 1$ in TYM , a basis $\langle \theta_\alpha \rangle, \alpha = 1, \dots, 16$ in S and $\langle v_i \rangle, i = 1, \dots, 10$ in V . Together they form a basis in $\langle e_\gamma, \theta_\alpha, v_i \rangle$ of $TYM + S + V \cong L$. The commutation relations in this basis are $[e_\gamma, e_{\gamma'}] = \sum_{\delta \geq 0} c_{\gamma\gamma'}^\delta e_\delta$, $[\theta_\alpha, e_\gamma] = \sum_k f_{\alpha\gamma}^\delta e_\delta$, $[v_i, e_\gamma] = \sum_k g_{ij}^k e_\gamma$, $[\theta_\alpha, \theta_\beta] = \Gamma_{\alpha\beta}^i v_i$, $[\theta_\alpha, v_i] = \sum_\delta h_{\alpha i}^\delta e_\delta$. The algebra $C^\bullet(L) \cong \text{Sym}\Pi L^* \cong \mathbb{C}[[t^1, \dots, t^{16}]] \otimes \Lambda[\xi^1, \dots, \xi^{10}] \otimes C^\bullet(TYM)$ has generators $\epsilon^\gamma, t^\alpha, \xi_i$ dual and opposite parity to

$e_\gamma, \theta_\alpha, v_i$. The differential can be written as

$$\begin{aligned} & c_{\gamma\gamma'}^\delta \epsilon^\gamma \epsilon^{\gamma'} \frac{\partial}{\partial \epsilon^\delta} + \\ & + f_{\alpha\gamma}^{\gamma'} t^\alpha \epsilon^\gamma \frac{\partial}{\partial \epsilon^{\gamma'}} + \\ & + g_{i\gamma}^{\gamma'} \xi^i \epsilon^\gamma \frac{\partial}{\partial \epsilon^{\gamma'}} + \\ & + r_{i\alpha}^\gamma \xi^i t^\alpha \frac{\partial}{\partial \epsilon^\gamma} + \Gamma_{\alpha\alpha'}^i t^\alpha t^{\alpha'} \frac{\partial}{\partial \xi_i} \end{aligned}$$

We identify $c_{\gamma\gamma'}^\delta \epsilon^\gamma \epsilon^{\gamma'} \frac{\partial}{\partial \epsilon^\delta}$ with the differential in $C^\bullet(TYM)$. All other terms define the desired L_∞ action.

To classify infinitesimal SUSY deformations of reduced YM theory it is sufficient to calculate Lie- Hochschild cohomology $HH_{\mathfrak{g}}(C^\bullet(YM), C^\bullet(YM))$. First of all we notice that one can use duality between $C^\bullet(YM)$ and $U(YM)$ to calculate Hochschild cohomology

$$HH(C^\bullet(YM), C^\bullet(YM)) = HH(U(YM), U(YM)) = H(YM, U(YM)) = H(YM, \text{Sym}(YM)). \quad (117)$$

Here we are using (113) and the relation between Hochschild cohomology of enveloping algebra $U(YM)$ and Lie algebra cohomology of YM as well as Poincaré -Birkhoff-Witt theorem. Analyzing the proof of (117) we obtain quasi-isomorphism between the complex $\text{cobar}C^\bullet(YM) \otimes C^\bullet(YM)$ that we are using in calculation of $HH(C^\bullet(YM), C^\bullet(YM))$ and the complex $\text{Sym}(\Pi YM)^* \otimes \text{Sym}YM$ with cohomology $H(YM, \text{Sym}(YM))$.

To calculate the Lie-Hochschild cohomology $HH_{\mathfrak{g}}(C^\bullet(YM), C^\bullet(YM))$ we should consider a complex $C^\bullet(\mathfrak{g}) \otimes \text{cobar}C^\bullet(YM) \otimes C^\bullet(YM)$ with the differential defined in (109). This complex is quasi-isomorphic to $C^\bullet(\mathfrak{g}) \otimes \text{Sym}(\Pi YM)^* \otimes \text{Sym}YM$ with appropriate differential.

Now we can notice that $C^\bullet(\mathfrak{g}) \otimes \text{Sym}(\Pi YM)^* = \mathbb{C}[[t^1, \dots, t^{16}]] \otimes \text{Sym}(\Pi YM)^* = \text{Sym}(\Pi L^*) = C^\bullet(L)$. Hence we can reassemble $C^\bullet(\mathfrak{g}) \otimes \text{Sym}(\Pi YM)^* \otimes \text{Sym}YM$ into $\text{Sym}(\Pi L^*) \otimes \text{Sym}(YM)$, which is isomorphic to $C^\bullet(L, U(YM))$.

We see that Lie- Hochschild cohomology $HH_{\mathfrak{g}}(C^\bullet(YM), C^\bullet(YM))$ with $\mathfrak{g} = \Pi \mathbb{C}^{16}$ classifying infinitesimal SUSY deformations in the reduced case is isomorphic to $H^\bullet(L, U(YM))$.

Lie-Hochschild cohomology $HH_{\mathfrak{g}}(C^\bullet(TYM), C^\bullet(TYM))$ where $\mathfrak{g} = \mathfrak{su}(n)$ govern SUSY deformations of unreduced SYM. Similar considerations permit us to prove that these cohomology are isomorphic to $H^\bullet(L, U(TYM))$.

The above statements agree with the theorems of Section 4 where we claimed that two-dimensional cohomology of L with coefficients in $U(YM)$ and in $U(TYM)$ correspond to SUSY deformations in reduced and unreduced cases. This claim was proven there in reduced case; the consideration of present section justify it in unreduced case.

We see that BV approach leads to wider class of SUSY deformations.³⁶ However, one can prove that all super Poincaré invariant deformations in the reduced case are covered by the constructions of Sections 4. The proof is based on the remark that the groups $H^i(L, U(YM))$ do not contain $\text{Spin}(10)$ -invariant elements for $i > 2$. (This remark can be derived from the considerations of Section 5.) Notice, however, that the group $H^i(L, U(TYM))$ contains $\text{Spin}(10)$ -invariant element for $i = 3$. This element is responsible for the super Poincaré invariant deformation of L_∞ action of supersymmetry, but it generates a trivial infinitesimal variation of action functional. However, corresponding formal deformation constructed in Section 7 can be non-trivial.

D L_∞ action of the supersymmetry algebra in the BV formulation

In Appendix C we have shown that one can construct an L_∞ action of SUSY algebra on bv . In this section we will give another proof of the existence of this action; we will show that this proof permits us to construct an L_∞ action that is compatible with invariant inner product on bv . We use the formalism of pure spinors in our considerations.

³⁶ One can modify the arguments of Section 4 to cover the additional deformations arising in BV formalism. The modification is based on consideration of A_∞ deformations of associative algebras $U(YM)$ and $U(TYM)$.

The pure spinor construction will be preceded by a somewhat general discussion of L_∞ -invariant traces.

Suppose that the tensor product $A \otimes \text{Sym}(\Pi \mathfrak{g})$ is furnished with a differential d which can be written as $d_A + d_{\mathfrak{g}} + q$, where d_A is a differential in A , $d_{\mathfrak{g}}$ is the Lie algebra differential (73) in $\text{Sym}(\Pi \mathfrak{g})$ and $q = \sum_{n \geq 1} \frac{1}{n!} c^{\alpha_1} \cdots c^{\alpha_n} q_{\alpha_1, \dots, \alpha_n}$ is the generating function of derivations $q_{\alpha_1, \dots, \alpha_n}$ that satisfies the analog of (107). We say that A is equipped with \mathfrak{g} -equivariant trace if there is a linear map

$$A \otimes \text{Sym}(\Pi \mathfrak{g}) \rightarrow \text{Sym}(\Pi \mathfrak{g})$$

which satisfies $p([a, a']) = 0$ and

$$p(Q_A + d_{\mathfrak{g}} + q)a = d_{\mathfrak{g}}pa$$

for every $a \in A \otimes \text{Sym}(\Pi \mathfrak{g})$. In the case when we have an ordinary action of a Lie algebra \mathfrak{g} on a differential graded Lie algebra A and a trace functional p is \mathfrak{g} -invariant, i.e. $p(la) = 0$ for any $l \in \mathfrak{g}$ and $a \in A$ then p is trivially a \mathfrak{g} -equivariant functional.

This construction provides us with an inner product $\langle a, b \rangle = p_{\mathfrak{g}}(ab)$ on A with values in $\text{Sym}(\Pi \mathfrak{g})$.

In pure spinor formalism the algebra $S \otimes C^\infty(\mathbb{R}^{10|16})$ is equipped with the differential D given by the formula (10) and the D -closed linear functional

$$p : S \otimes C^\infty(\mathbb{R}^{10|16}) \rightarrow \mathbb{C} \tag{118}$$

It is defined on elements that decay sufficiently fast at the space-time infinity. The functional p splits into a tensor product of translation-invariant volume form vol on \mathbb{R}^{10} and a functional $p_{\text{red}} : S \otimes C^\infty(\mathbb{R}^{0|16}) \rightarrow \mathbb{C}$.

The super-symmetries generators are

$$\theta_\alpha = \frac{\partial}{\partial \psi^\alpha} + \Gamma_{\alpha\beta}^i \psi^\beta \frac{\partial}{\partial x^i}.$$

The functional p_{red} is $\text{Spin}(10)$ -invariant. Also it can be characterised as the only nontrivial $\text{Spin}(10)$ -invariant functional on $S \otimes C^\infty(\mathbb{R}^{0|16})$. This follows

from simple representation theory for $\text{Spin}(10)$. This fact enables us to construct an "explicit" formula for p_{red} . The projection

$$\mathbb{C}[\lambda^1, \dots, \lambda^{16}] \xrightarrow{k} S$$

commutes with the action of $\text{Spin}(10)$. A simple corollary of representation theory is that k has a unique linear $\text{Spin}(10)$ -equivariant splitting k^{-1} . Using this splitting we can identify elements of S with Γ -traceless elements in $\mathbb{C}[\lambda^1, \dots, \lambda^{16}]$. Let us define a $\text{Spin}(10)$ -invariant differential operator on $\mathbb{C}[\lambda^1, \dots, \lambda^{16}] \otimes \Lambda[\psi^1, \dots, \psi^{16}]$ by the formula

$$P = \Gamma_m^{\alpha\beta} \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \psi^\beta} \Gamma_n^{\gamma\delta} \frac{\partial}{\partial \lambda^\gamma} \frac{\partial}{\partial \psi^\delta} \Gamma_k^{\epsilon\epsilon} \frac{\partial}{\partial \lambda^\epsilon} \frac{\partial}{\partial \psi^\epsilon} \Gamma_{mnk}^{\mu\nu} \frac{\partial}{\partial \psi^\mu} \frac{\partial}{\partial \psi^\nu} \quad (119)$$

We define $p_{\emptyset}(a)$ as $Pa|_{\lambda, \psi=0}$.

One of the properties of p_{\emptyset} is that it is D -closed

$$p_{\emptyset}(Da) = 0.$$

It is not, however, invariant with respect to the action of supersymmetries. It satisfies a weaker condition

$$p_{\emptyset}(\theta_\alpha a) = p_\alpha(Da)$$

The generating function technique that was used for formulation of L_∞ action in the BV formulation can be used here. We have even ghosts t^1, \dots, t^{16} and odd ghosts ξ^1, \dots, ξ^{10} . We define the total L_∞ action operator D_∞ as the sum

$$D_\infty = D + t^\alpha \theta_\alpha + \xi^i \frac{\partial}{\partial x^i} + d_{\text{susy}} \quad (120)$$

The condition $D_\infty^2 = 0$ is equivalent to the standard package of properties of the pure spinor BV differential and supersymmetries.

We will construct a generating function of functionals p that satisfies equation

$$p(a) = d_{\text{susy}} p(a) + \text{exact terms} \quad (121)$$

We define a formal perturbation p of P as a composition

$$p = \sum_{k \geq 0} \frac{1}{k!} P \left(t^\alpha \frac{\partial}{\partial \lambda^\alpha} \right)^k \quad (122)$$

Then

$$p_{\varnothing}(a) \text{ is equal to } p(a)|_{t, \psi=0}$$

The L_∞ -invariant trace functional can be defined as

$$p_{\text{susy}}(a) = \int p(a) \text{vol}$$

The (ghost dependent) inner product corresponding to this trace is also L_∞ invariant, hence we have proved that the L_∞ action is a Hamiltonian action.

E Calculation of hypercohomology

To justify the calculations of Section 5 we should check that the embedding $\mathcal{W}^* \rightarrow \mathcal{YM}^\bullet$ and the embedding $\text{Sym}^i(\mathcal{W}^*) \rightarrow \text{Sym}^i(\mathcal{YM})$ are quasi-isomorphisms. In other words, we should prove that these homomorphisms induce isomorphisms of hypercohomology. We should prove also similar results for embedding $\mathcal{W} \rightarrow \mathcal{TYM}$ and embedding $\text{Sym}^i(\mathcal{W}) \rightarrow \text{Sym}^i(\mathcal{YM})$.

We will start with some general considerations. As we have noticed in Section 5, there are two spectral sequences that can be used in calculation of hypercohomology of the complex of vector bundles \mathcal{N}_I^\bullet . Here we will use the second one (with $E_2 = H^i(H^j(\Omega(\mathcal{N}_I), d_e), \bar{\partial})$).

First of all we will consider the modules N where $N = L, YM$ or TYM , corresponding differential vector bundles $\mathcal{N} = \mathcal{L}, \mathcal{YM}, \mathcal{TYM}$ and differential P -modules N_P obtained as fibers of these bundles over the point $\lambda_0 \in \mathcal{Q}$. The differential on the module N_P is obtained as restriction of the differential d_e on vector bundle \mathcal{N} and will be denoted by the same symbol.

Let us start with calculation of the cohomology of the module $U(L)_P$.

Proposition 34 $H^i(U(L)_P, d_e) = HH^i(S, \mathbb{C}_{\lambda_0})$

Here one-dimensional S -bimodule \mathbb{C}_{λ_0} is obtained by specialization at $\lambda_0 \in C\mathcal{Q}$ with coordinates λ_0^α . In more details the left and right actions of polynomial $f(\lambda)$ on generators $a \in \mathbb{C}_{\lambda_0}$ is given by the formula $f(\lambda) \times a = f(\lambda_0)a$

Proof. This is a direct application of Proposition 7 where $N = \mathbb{C}_{\lambda_0}$ and $A = \mathcal{S}, A^! = U(L)$. ■

To calculate RHS in Proposition 34 we use the following statement that can be considered as a weak form of Hochschild-Kostant-Rosenberg theorem (see [8]):

Proposition 35 *Suppose A is a ring of algebraic functions on affine algebraic variety. Let \mathbb{C}_x denote a one-dimensional bimodule, corresponding to a smooth point x . Then $HH^i(A, \mathbb{C}_x) = \Lambda^i(T_x)$, where T_x is the tangent space at x .*

Corollary 36 $HH^i(S, \mathbb{C}_{\lambda_0}) = H^i(U(L)_P, d_e) = H^i(\text{Sym}(L)_P, d_e) = \Lambda^i(T_{\lambda_0})$, where T_{λ_0} is the tangent space to $C\mathcal{Q}$ at the point $\lambda_0 \neq 0$. It follows from this that $H^i(\text{Sym}^i(L)_P, d_e) = \Lambda^i(T_{\lambda_0})$ and $H^j(\text{Sym}^i(L)_P, d_e) = 0, i \neq j$. In particular, for $i = 1$ we obtain $H^1(L_P) = T_{\lambda_0}, H^j(L_P) = 0$ if $j > 1$.

The corollary follows from Proposition 35 because $C\mathcal{Q}$ is a smooth homogeneous space away from $\lambda = 0$.

Recall that Lie algebra L as a vector space is equal to $L^1 + YM$. The action of the differential d_e is P -covariant. This fact together with the information about the cohomology of L_P permits us to calculate the action of d_e on L_P and on YM_P .

Recall that

$$L_P = L^1 \otimes \mu_{-1} + L^2 + L^3 \otimes \mu_1 + \dots$$

We describe the differential d_e on $L^1 \otimes \mu_{-1}$ using decomposition (32-33).

It follows from (32) that $L^1 \otimes \mu_{-1}$ has W^* as factor -representation, i.e there exists a surjective homomorphism $\phi : L^1 \otimes \mu_{-1} \rightarrow W^*$. We conclude from Schur's lemma that d_e maps $L^1 \otimes \mu_{-1}$ onto $W^* \subset L^2$ and coincides with ϕ up to a constant factor. From the information about the cohomology of L_P we infer that the constant factor does not vanish. Taking into account that

the $H^i(L_P, d_e) = 0$ for $i > 1$ we obtain that the complex $L^1/\text{Ker}d_e \rightarrow L^2 \rightarrow \dots$ is acyclic. If we truncate $L^1/\text{Ker}d_e$ term, the resulting complex will have cohomology equal to $d_e(L^1/\text{Ker}d_e) = W^*$. This proves that the embedding $W^* \subset YM_P$ is a quasi-isomorphism.

To derive from this statement that the embedding of vector bundles $\mathcal{W}^* \subset \mathcal{YM}$ generates isomorphism of hypercohomology we notice that this embedding induces a homomorphism of spectral sequences calculating the hypercohomology. It is easy to check that the above statement implies isomorphism of E_2 terms, hence isomorphism of hypercohomology.

From Künneth theorem we can conclude that the embedding $\text{Sym}^j W \subset \text{Sym}^j YM_P$ is a quasi-isomorphism; using spectral sequences we derive isomorphism of hypercohomology of corresponding complexes of vector bundles.

We can give a similar analysis of the complex \mathcal{TYM}^\bullet . Indeed we have a short exact sequence of complexes

$$0 \rightarrow \mathcal{TYM}^\bullet \rightarrow \mathcal{YM}^\bullet \rightarrow \mathcal{L}^2 \rightarrow 0,$$

where \mathcal{L}^2 is a trivial vector bundle over \mathcal{Q} with a fiber L^2 . The short exact sequence gives rise to short exact sequence of corresponding P -modules and to a long exact sequence of their cohomology:

$$\begin{aligned} 0 \rightarrow H^0(YM_P, d_e) \rightarrow L^2 \rightarrow H^1(TYM_P, d_e) \rightarrow 0 \\ H^i(TYM_P, d_e) = H^i(YM_P, d_e) \quad i \geq 2 \end{aligned} \tag{123}$$

By definition $H^0(TYM_P, d_e) = 0$. Taking into account quasi-isomorphism between W^* and YM_P we get an exact sequence of $\text{Spin}(10)$ -modules

$$0 \rightarrow W^* \rightarrow L^2 \rightarrow H^1(TYM_P, d_e) \rightarrow 0$$

It follows from the decomposition (33) that there is only one $\text{Spin}(10)$ -equivariant embedding of $\text{Spin}(10)$ -modules: $W^* \rightarrow L^2$. Also the module L^2/W^* is isomorphic to W . From this we conclude that $H^1(TYM_P, d_e)$ is isomorphic to W . Due to isomorphisms (123) the complex TYM_P has no higher cohomology.

We see that the embedding $W \rightarrow TYM_P$ is a quasi-isomorphism. Again using spectral sequences we obtain that the embedding

$$\mathcal{W} \rightarrow \mathcal{TYM}^\bullet \tag{124}$$

generates an isomorphism of hypercohomology.

References

- [1] E. A. Bergshoeff, A. Bilal, M. de Roo, A. Sevrin *Supersymmetric non-abelian Born-Infeld revisited* JHEP 0107 (2001) 029, hep-th/0105274
- [2] N. Berkovits *Covariant Quantization of the Superparticle Using Pure Spinors* JHEP 0109 (2001) 016, hep-th/0105050
- [3] N. Berkovits *Explaining the Pure Spinor Formalism for the Superstring* JHEP0801:065,2008, arXiv:0712.0324
- [4] N. Berkovits, N. Nekrasov *The character of pure spinors* hep-th/0503075
- [5] R Bott *Homogeneous vector bundles* Annals of Mathematics, 1957
- [6] D. Burghilea, Z. Fiedorowicz and W. Gajda *Adams operations in Hochschild and cyclic homology of de Rham algebra and free loop spaces* K-Theory 4 (1991), no. 3, 269–287.
- [7] A. Collinucci, M. de Roo, M. G. C. Eenink *Supersymmetric Yang-Mills theory at order α'^3* JHEP 0206 (2002) 024
- [8] A. Connes *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994. xiv+661 pp. IS: 0-12-185860-X
- [9] B. L. Feigin, B. L. Tsygan *Cyclic homology of algebras with quadratic relations, universal enveloping algebras and group algebras*. K-theory, arithmetic and geometry (Moscow, 1984–1986), 210–239, Lecture Notes in Math., 1289, Springer, Berlin, 1987. Consultants Bureau, New York, 1986. (1988), no. 2, 221–251.

- [10] M. Kontsevich, private communication
- [11] B. Keller *Invariance and localization for cyclic homology of DG algebras.* J. Pure Appl. Algebra 123 (1998), no. 1-3, 223–273.
- [12] B. Keller *Derived invariance of higher structures on the Hochschild complex* preprint available at <http://people.math.jussieu.fr/~keller/publ/dih.pdf>
- [13] J.-L. Loday *Cyclic homology.* Grundlehren der Mathematischen Wissenschaften, 301. Springer-Verlag, Berlin, 1998.
- [14] S. Mac Lane *Homology.* Die Grundlehren der mathematischen Wissenschaften, Bd. 114 Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg 1963
- [15] M. Movshev *Yang-Mills theories in dimensions 3,4,6,10 and Bar-duality* hep-th/0503165
- [16] M. Movshev *Cohomology of Yang-Mills algebras* J. Noncommut. Geom. 2 (2008), no. 3, 353–404
- [17] M. Movshev *Deformation of maximally supersymmetric Yang-Mills theory in dimensions 10. An algebraic approach* arXiv:hep-th/0601010v1
- [18] M. Movshev, A.Schwarz *On maximally supersymmetric Yang-Mills theories* hep-th/0311132
- [19] M. Movshev, A.Schwarz *Algebraic structure of Yang-Mills theory* hep-th/0404183
- [20] M. Movshev *Deformation of maximally supersymmetric Yang-Mills theory in dimensions 10. An algebraic approach* hep-th/0601010
- [21] N. Berkovits, N. Nekrasov *The Character of Pure Spinors* Lett.Math.Phys. 74 (2005) 75-109 arXiv:hep-th/0503075v1.
- [22] M. Penkava, A.Schwarz *A_∞ Algebras and the Cohomology of Moduli Spaces* hep-th

- [23] S. B. Priddy, *Koszul resolutions* Trans. Amer. Math. Soc., 152, No. 1, 39–60 (1970).